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P_m-SATURATED GRAPHS WITH MINIMUM SIZE

Abstract. By P_m we denote a path of order m. A graph G is said to be $P_m - saturated$ if G has no subgraph isomorphic to P_m and adding any new edge to G creates a P_m in G. In 1986 L. Kászonyi and Zs. Tuza considered the following problem: for given m and n find the minimum size $sat(n; P_m)$ of P_m-saturated graph and characterize the graphs of $Sat(n; P_m)$ – the set of P_m-saturated graphs of minimum size. They have solved this problem for $n \geq a_m$ where $a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if} \quad m = 2k, k > 2 \\ 2^{k+1} - 2 & \text{if} \quad m = 2k+1, k \geq 2 \end{cases}$. We define $b_m = \begin{cases} 3 \cdot 2^{k-2} & \text{if} \quad m = 2k, k \geq 3 \\ 3 \cdot 2^{k-1} - 1 & \text{if} \quad m = 2k+1, k \geq 3 \end{cases}$ and give $sat(n; P_m)$ and $Sat(n; P_m)$ for $m \geq 6$ and $b_m \leq n < a_m$.

Keywords: graph, saturated graph, extremal graph.

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1. INTRODUCTION

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively, $|G|, e(G)$ the order and the size of G and $d_G(v)$ the degree of $v \in V(G)$. For simplicity we shall suppose $|G| = n$. By P_m we denote the path of order m and by K_r the complete graph on r vertices. For vertex disjoint graphs G and H we denote by $G * H$ the graph with vertex set $V(G * H) = V(G) \cup V(H)$, and edge set $E(G * H) = E(G) \cup E(H) \cup \cup\{xy : x \in V(G), y \in V(H)\}$. The union $G \cup H$ of graphs G and H is defined by $V(G \cup H) = V(G) \cup V(H)$, $E(G \cup H) = E(G) \cup E(H)$, and we shall always suppose that the components of the union are vertex disjoint. A vertex of a graph is called penultimate if it is a neighbour of a vertex of degree 1 (*pendent vertex*).

A graph G is H-saturated if there is no subgraph of G isomorphic to H and adding any new edge e to G creates H. We shall denote by $G \cup e$ the graph obtained from G by the addition of the edge e, supposing that the two end vertices of e are in G. We define also:

$$sat(n; P_m) = \min\{e(G) : |G| = n, G \text{ is } P_m\text{-saturated}\},$$
$$Sat(n; P_m) = \{G : |G| = n, e(G) = sat(n; P_m), G \text{ is } P_m\text{-saturated}\}.$$

The best general reference here is [1]. The first and well known result considering H-saturated graphs with minimum size is the following theorem of P. Erdös, A. Hajnal and J. W. Moon, [6].

Theorem 1. *The minimum size of K_m-saturated graph of order n*

$$sat(n; K_m) = \binom{m-2}{2} + (m-2)(n-m+2) \ , \quad (n \geq m \geq 2)$$

and the only K_m-saturated graph of size $sat(n; K_m)$ is $K_{m-2} * \overline{K}_{n-m+2}$.

P_m-saturated graphs of order n with minimum size for m small have been characterized by L. Kászonyi and Zs. Tuza in [7].

Theorem 2 (L. Kászonyi and Zs. Tuza).

$$sat(n; P_3) = \lfloor \tfrac{n}{2} \rfloor \ ;$$
$$Sat(n; P_3) = \begin{cases} kK_2 & \textit{if} \quad n = 2k, \\ kK_2 \cup K_1 & \textit{if} \quad n = 2k+1; \end{cases}$$
$$sat(n; P_4) = \begin{cases} k & \textit{if} \quad n = 2k, \\ k+2 & \textit{if} \quad n = 2k+1; \end{cases}$$
$$Sat(n; P_4) = \begin{cases} kK_2 & \textit{if} \quad n = 2k, \\ (k-1)K_2 \cup K_3 & \textit{if} \quad n = 2k+1; \end{cases}$$
$$sat(n; P_5) = n - \lfloor \tfrac{n-2}{6} \rfloor - 1 \quad (\textit{for } n \geq 6).$$

Let us suppose that $m \geq 5$ is an integer. Then A_m is the following tree. All non pendent vertices of A_m have their degree equal to three. If $m = 2k$, $k \geq 3$, then A_m has one center and k levels. If $m = 2k+1$, $k \geq 2$, then A_m has two centers v_1, v_2 and each component of $G - \{v_1, v_2\}$ has $k-1$ levels. Observe that $|A_m| = a_m$, where

$$a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if} \quad m = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if} \quad m = 2k+1, k \geq 2. \end{cases}$$

For $m > n$ the only P_m-saturated graph of order n is K_n. Hence K_1 and K_2 are the only P_m-saturated trees when $m > n$. It is easy to see that the trees A_m are P_m-saturated. Moreover, every graph in which every component is isomorphic to A_m or is obtained from A_m by multiplying some branches, is P_m-saturated. In [7] Kászonyi and Tuza proved much more, namely the following theorem.

Theorem 3 (L. Kászonyi and Zs. Tuza). *Let $m \geq 5$. If T is a P_m-saturated tree then either $A_m \subset T$ or $T = K_i$ with $i \in \{1, 2\}$.*

Theorem 3 implies easily:

Corollary 1 (L. Kászonyi and Zs. Tuza). *Let $m \geq 6$ and let $G = (V; E)$ be a P_m-saturated graph with minimum size such that $|G| = n \geq a_m$. Then $|E(G)| = n - \lfloor \frac{n}{a_m} \rfloor$ and every component of G is a tree containing A_m.*

In particular, for $m \geq 6$ and $n \geq a_m$ we have $sat(n; P_m) = n - \lfloor \frac{n}{a_m} \rfloor$. It is natural to try to find P_m-saturated graphs with minimum size for $n < a_m$. In [3] we give a general upper bound for $sat(n; P_m)$ for some m and n.

Theorem 4. *Let $n \geq m$.*

1. *For m even, $m \in \{22, 30, 38, 40, 42, 46, 48, 50\}$ or $m \geq 54$*

$$sat(n; P_m) \leq n + \frac{m}{2} - 1.$$

2. *For m odd, $m \in \{23, 31, 39, 41, 43, 47, 49, 51\}$ or $m \geq 55$*

$$sat(n; P_m) \leq \frac{3n}{2}.$$

In section 2 we introduce the notation, look more closley at P_m-saturated graphs with exactly one cycle and prove that $sat(n; P_m) = n$ for $7 \leq m$, $b_m \leq n < a_m$ where

$$b_m = \begin{cases} 3 \cdot 2^{k-2} & \text{if} \quad m = 2k, k \geq 3 \\ 3 \cdot 2^{k-1} - 1 & \text{if} \quad m = 2k+1, k \geq 3 \end{cases} \tag{1}$$

and we shall give $Sat(n; m)$ for $6 \leq m \leq n, b_m \leq n < a_m$. Note that, by Theorem 3, every P_m-saturated graph G of order $n \geq m$ such that $5 \leq n < a_m$, contains at least one cycle. Hence, it is clear that the minimum size of a connected P_m-saturated graph G of order $|G| < a_m$ is at least $|G|$.

The P_m-saturated graphs were considered by P. Erdös and T. Gallai in [5] who proved that the maximum size of P_m-saturated graphs of order n is equal to $\frac{n(m-2)}{2}$ when $n \equiv 0 \bmod (m-1)$ and then the extremal graph is isomorphic with $\frac{n}{m-1} K_{m-1}$. The maximum size of hamiltonian path saturated graphs of order n is equal to $\binom{n-1}{2}$ for $n \geq 2$ (see also [2, 4, 8, 9]).

2. P_m-SATURATED GRAPHS WITH EXACTLY ONE CYCLE

The section starts with eight easy claims in which we shall assume that n and m are integers, $n \geq m \geq 6$ and G is a connected P_m-saturated graph with exactly one non trivial cycle $C = (c_1, \ldots, c_k, c_1)$, $k \geq 3$. We denote by x_l, $l = 1, 2, \ldots, p_1$, the neighbours of c_1 such that $x_l \notin V(C)$, by y_l, $l = 1, 2, \ldots, p_2$, the neighbours of c_2 such that $y_l \notin V(C)$, and by z_l, $l = 1, 2, \ldots, p_3$, the neighbours of c_3 such that $z_l \notin V(C)$. Let $u \in V(G)$. For simplicity we shall denote by $L_G(u)$ — the order of the longest path in G containing u, and by $L_v(u)$ — the order of the longest path in G starting from u and containing v where $v \in N_G(u)$. We shall denote for $i = 1, 2, 3$ by L_{ij} — the order of the longest path in G starting from c_i, $i = 1, 2, 3$, not containing any other vertex from $V(C)$, and containing respectively x_j for $i = 1$, y_j for $i = 2$, and z_j for $i = 3$, $j = 1, \ldots, p_i$. For $j = 1$ we denote $L_{i1} = L_i$, for $i = 1, 2 \ldots, k$. For convenience we shall assume:

$$\begin{aligned} &L_1 \geq L_{12} \geq \cdots \geq L_{1p_1}, \\ &L_2 \geq L_{22} \geq \cdots \geq L_{2p_2}, \\ &L_3 \geq L_{32} \geq \cdots \geq L_{3p_3}. \end{aligned} \tag{2}$$

By $|\ldots v_1 v_2 \ldots v_r \ldots|$ we shall denote the order of a longest path containing the segment $v_1 v_2 \ldots v_r$.

Mean set M_G is the set of all central vertices of paths of order $m - 1$ in G. Every $v \in M_G$ will be called *mean vertex*.

Claim 1. $|C| = 3$.

Proof. To obtain a contradiction suppose that $|C| \geq 4$. Without loss of generality we may assume $L_2 = \min\{L_i : i = 1, 2, \ldots, k\}$. The idea of the proof is to show it is impossible to create P_m by connecting c_1 with c_3. Observe that $|\ldots x_1 c_1 c_3 z_1 \ldots| < |\ldots x_1 c_1 c_2 c_3 z_1 \ldots|$ and $|\ldots x_1 c_1 c_3 c_4 \ldots| < |\ldots x_1 c_1 c_2 c_3 c_4 \ldots|$. Hence the only possibility to create P_m in $G \cup \{c_1 c_3\}$ is to use the path $\ldots y_1 c_2 c_3 c_1 c_k \ldots$ or $\ldots y_1 c_2 c_1 c_3 c_4 \ldots$. Thus we have either $m \leq |\ldots y_1 c_2 c_3 c_1 c_k \ldots| \leq |\ldots z_1 c_3 c_2 c_1 c_k \ldots|$ or $m \leq |\ldots y_1 c_2 c_1 c_3 c_4 \ldots| \leq |\ldots x_1 c_1 c_2 c_3 c_4 \ldots|$. A contradiction. □

Claim 2. *Mean set of G is either a singleton or a set of two adjacent vertices or else the set $C = \{c_1, c_2, c_3\}$ of the vertices of the cycle C.*

Proof. It is easy to see that the mean vertices are adjacent. The claim follows. □

Claim 3. *Let $C = (c_1, c_2, c_3, c_1)$. If $d_G(c_1) = 2$ and $d_G(c_2) \neq 2$ then $d_G(c_3) = 2$.*

Proof. The proof will be divided into two steps.

By contradiction suppose first that $d_G(c_3) = 3$. By adding an edge $c_1 z_1$ we create P_m in $G \cup \{c_1 z_1\}$ which contains the segment $z_1 c_1 c_3 c_2$ or $c_3 z_1 c_1 c_2$ (in case $L_3 = 2$). But replacing $z_1 c_1 c_3 c_2$ ($c_3 z_1 c_1 c_2$) by $z_1 c_3 c_1 c_2$ we obtain P_m which does not contain the new edge $c_1 z_1$, a contradiction.

So we may suppose now that $d_G(c_3) \geq 4$. It is easy to observe that without loss of generality we may assume $L_2 \geq L_3$. We next show that by adding the edge $z_2 c_2$ we will not create P_m. Observe first that $y_1 c_2 z_2 c_3 z_1$ can be replaced by $y_1 c_2 c_1 c_3 z_1$. The only way to create P_m in $G \cup \{z_2 c_2\}$ is to use the path $\ldots z_2 c_2 c_1 c_3 z_1 \ldots$ Since $L_2 \geq L_3$ and (2) we have $|\ldots z_2 c_2 c_1 c_3 z_1 \ldots| \leq |\ldots z_1 c_3 c_1 c_2 y_1 \ldots|$, a contradiction. □

Claim 4. *Let $C = (c_1, c_2, c_3, c_1)$. If $d_G(c_1) = 2$ and $d_G(c_3) = 2$ then $d_G(c_2) \geq 4$.*

Claim 5. *If $v \in V(G)$ is such a vertex that $d_G(v) = 2$ then $v \in V(C)$.*

Claim 6. *Let $u \in V(G) - V(C)$, $d_G(u) \neq 1$, $N_G(u) = \{u_1, u_2, \ldots, u_p\}$ and $L_{u_1}(u) \geq > \geq L_{u_2}(u) \geq \cdots \geq L_{u_p}(u)$. Then:*

1. $L_{u_2}(u) = L_{u_3}(u)$,
2. $L_G(u) = m - 1$.

Proof. Suppose that the first assertion of the Claim 6 is false. Then $L_{u_2}(u) > L_{u_3}(u)$. Since $u \notin V(C)$, we have $u_1u_2 \notin E(G)$. Observe first that to create P_m in $G \cup \{u_1u_2\}$ we have to use $u_1u_2uu_3$. We obtain:

$$m \leq L_{u_1}(u) + L_{u_3}(u) \tag{3}$$

Since G is P_m- saturated we have $L_G(u) = L_{u_1}(u) + L_{u_2}(u) - 1 < m$ so

$$L_{u_1}(u) + L_{u_2}(u) \leq m \tag{4}$$

We conclude from (3) and (4) that $L_{u_2}(u) \leq L_{u_3}(u)$ and finally get a contradiction. To prove the second part of the Claim 6 we use inequalities (3), (4) and equality $L_{u_2}(u) = L_{u_3}(u)$. □

Claim 7. *Let $C = (c_1, c_2, c_3, c_1), d_G(c_i) \neq 2$, for $i = 1, 2, 3$ and $L_1 \leq L_2 \leq L_3$. Then:*

1. $L_1 = L_2$,
2. *If $L_1 \neq L_3$ then $d_G(c_3) \geq 4$.*

Proof. In the first part of the claim it is easily seen that it would be impossible to create P_m in $G \cup \{c_2z_1\}$ if $L_1 < L_2$. For similar reasons we have $d_G(c_3) \geq 4$. □

Claim 8.

1. *If there is in G a vertex u such that $L_G(u) < m - 1$ then $d_G(u) \leq 2$.*
2. *If $C = (c_1, c_2, c_3, c_1)$, $d_G(c_1) = d_G(c_2) = 2$ and $L_G(c_1) < m - 1$ then $d_G(c_3) \geq 5$.*
3. *If $C = (c_1, c_2, c_3, c_1)$, $d_G(c_1) = d_G(c_2) = 2$, $L_G(c_1) = m - 1$ then $L_G(c_2) = L_G(c_3) = m - 1$.*

Proof. We give the proof for first part only, the proofs of the others parts of the claim are left to the reader. Let $u \in V(G)$ and $L_G(u) < m - 1$ and suppose that $d_G(u) > 2$. By Claim 6, we have $u \in V(C)$. Observe that if the conditions $d_G(c_i) > 2$ and $L_G(c_i) < m-1$ hold for exactly two vertices c_i, $i = 1, 2, 3$, then for the remaining vertex c_j we have $d_G(c_j) > 2$ and $L_G(c_j) = m - 1$ (otherwise we have exactly one vertex c_i of degree 2 contrary to Claim 3). If $d_G(c_1) > 2$ and $L_G(c_1) < m - 1$ and $d_G(c_i) = 2$ or $L_G(c_i) = m - 1$ for $i = 2, 3$, then $L_G(c_1) = L_G(c_2)$, a contradiction. Thus the proof falls into three cases.

Case 1. $d_G(c_i) > 2$, $L_G(c_i) < m-1$ for $i = 1, 2, 3$.
From Claim 7 follows it is impossible that only one of the vertex x_1, y_1, z_1 have his degree equal to 1. If exactly two of vertices x_1, y_1, z_1 have his degree equal to 1, say $d_G(x_1) = d_G(y_1) = 1$, then by Claim 7 we have $d_G(c_3) \geq 4$. It is easy to see that if $d_G(z_1) \neq 1$ then $d_G(z_2) \neq 1$. From Claim 6 follows $L_G(z_1) = L_G(z_2) = m-1$. Observe that there exist vertex $u_1 \in N_G(z_1)$ and $u_2 \in N_G(z_2)$ and $u_1, u_2 \notin V(C)$ such that $L_{u_1}(z_1) \geq \lceil \frac{m-1}{2} \rceil$ and $L_{u_2}(z_2) \geq \lceil \frac{m-1}{2} \rceil$. We obtain the path containing segment $\ldots u_1 z_1 c_3 z_2 u_2 \ldots$ such that $|\ldots u_1 z_1 c_3 z_2 u_2 \ldots| \geq m-1$, a contradiction. If $d_G(x_1) = d_G(y_1) = d_G(z_1) = 1$ then there is no vertex u with $L_G(u) < m-1$ (observe that $m = 6$). So we may suppose that $d_G(x_1) > 1$, $d_G(y_1) > 1$, $d_G(z_1) > 1$ (note that since x_1, y_1, z_1 are not on the cycle we have then $d_G(x_1) \geq 3$, $d_G(y_1) \geq 3$, $d_G(z_1) \geq 3$). Let $v_1, v_2 \in N_G(x_1)$ and $v_1, v_2 \notin V(C)$, $w_1, w_2 \in N_G(y_1)$ and $w_1, w_2 \notin V(C)$. We shall assume $L_{v_1}(x_1) \geq L_{v_2}(x_1)$ and $L_{w_1}(y_1) \geq L_{w_2}(y_1)$. By Claim 6 we have $L_G(x_1) = L_G(y_1) = m-1$. We must have $L_{v_1}(x_1) \geq \lceil \frac{m-1}{2} \rceil$ and $L_{w_1}(y_1) \geq \lceil \frac{m-1}{2} \rceil$. The path $\ldots v_1 x_1 c_1 c_3 c_2 y_1 w_1 \ldots$ has the order at least m, which is impossible.

Case 2. $d_G(c_i) > 2$, $L_G(c_i) < m-1$ for $i = 1, 2$ and $L_G(c_3) = m-1$.
The proof is similar to the proof of the former case.

Case 3. $d_G(c_1) > 2$, $L_G(c_1) < m-1$ and $L_G(c_i) = m-1$ for $i = 2, 3$.
This case is easy, and we leave it to the reader. □

Construction of the family $\mathcal{R}_m$. For every integer $m \geq 6$ we shall define a family of connected P_m-saturated graphs containing exactly one cycle. Let us consider the tree $A_m = (V'; E')$. $\mathcal{R}_m$ is the family of graphs which may be obtained in the following way. Let $u \in V'$ be a vertex of A_m such that $d_{A_m}(u) = 3$ and let $N_{A_m}(u) = \{x, y, z\}$. If $d_{A_m}(x) \neq 1$, $d_{A_m}(y) \neq 1$ and $d_{A_m}(z) \neq 1$ then denote by x_1, x_2 the neighbours of x such that $\{x_1, x_2\} = N_{A_m}(x) \backslash \{u\}$, by y_1, y_2 the neighbours of y such that $\{y_1, y_2\} = N_{A_m}(y) \backslash \{u\}$ and by z_1, z_2 the neighbours of z such that $\{z_1, z_2\} = N_{A_m}(z) \backslash \{u\}$. First define $A_m(u)$ by removing u with incident edges and adding the edges xy, xz, yz. We shall consider three cases.

Case 1. u is the only central vertex of A_m (in particular m is even).
Define R_m^1 by removing from $A_m(u)$ the edges xx_1, yy_1, zz_1 and the components containing vertices x_1, y_1, z_1 (see Fig. 1).

Case 2. $d_{A_m}(u) = 3$ and u is neither the only central nor penultimate vertex of A_m.
Without loss of generality we may suppose that $L_{x_1}(x) = L_{x_2}(x) = L_{y_1}(y) = $ $= L_{y_2}(y) < L_{z_1}(z)$. Then define $R_m^2(u)$ deleting from $A_m(u)$ the edges xx_1, yy_1 and the components containing x_1 and y_1 (see Fig. 2). By $\mathcal{R}_m^2$ we denote the set of all the graphs $R_m^2(u)$.

Case 3. u is a penultimate vertex of A_m.
Let $v, w \in N_{A_m}(x_1), d_{A_m}(v) = d_{A_m}(w) = 1$. Define R_m^3 deleting from $A_m(u)$ the vertices v, w with incident edges. Put $\mathcal{R}_m = \{R_m^1\} \cup \mathcal{R}_m^2 \cup \{R_m^3\}$ (see Fig. 3). Note, that the least order of a graph $H \in \mathcal{R}_m$ is equal to b_m defined by the formula (1).

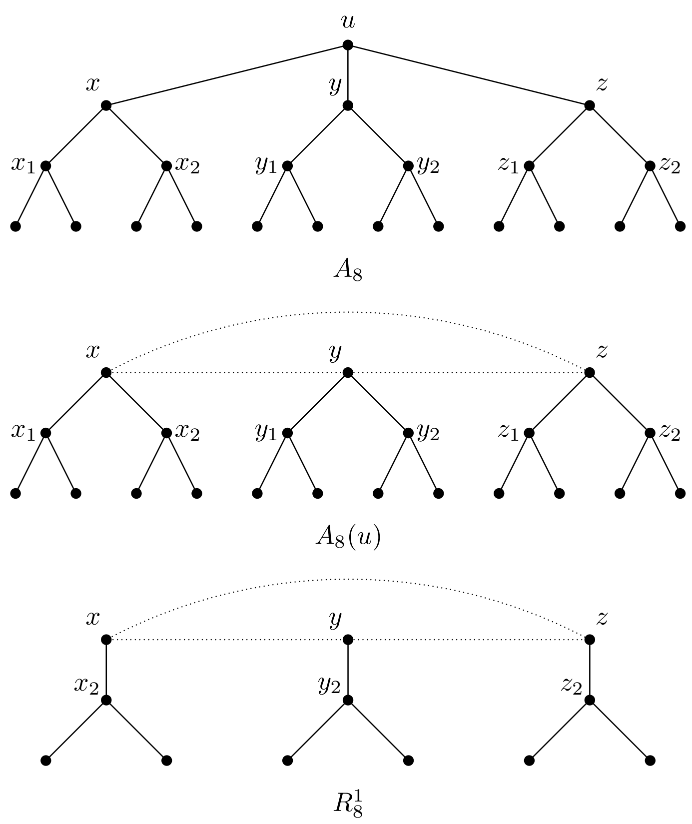


Fig. 1

Let H' be a graph obtained from R_6^1 by addition of a number of pendent neighbours of the vertices of the triangle. Then $G = H' \cup K_2$ is a P_6-saturated graph such that $|E(G)| = |G| - 1$.

Note that, for $m \geq 6$, the union of K_1 and a graph $H \in \mathcal{R}_m$ is never P_m-saturated.

Lemma 1. *Let n and m be integers such that $6 \leq m \leq n$ and $b_m \leq n < a_m$. Then the minimum size of a connected P_m-saturated graph of order n is equal to n.*

Proof. Since $6 \leq n < a_m$, by Theorem 3, there is no P_m-saturated tree of order n. On the other hand, for every $n \geq b_m$ there is an unicyclic connected graph G of order n which is P_m-saturated (in fact, G may be obtained from a graph $H \in \mathcal{R}_m$ by addition of a suitable number of pendent vertices). The lemma follows. $\square$

Lemma 2. *Let G be a connected unicyclic P_m-saturated graph of order n, $6 \leq m \leq$ $\leq n < a_m$, such that there are two vertices a, b of degree 2. Then G contains R_m^3.*

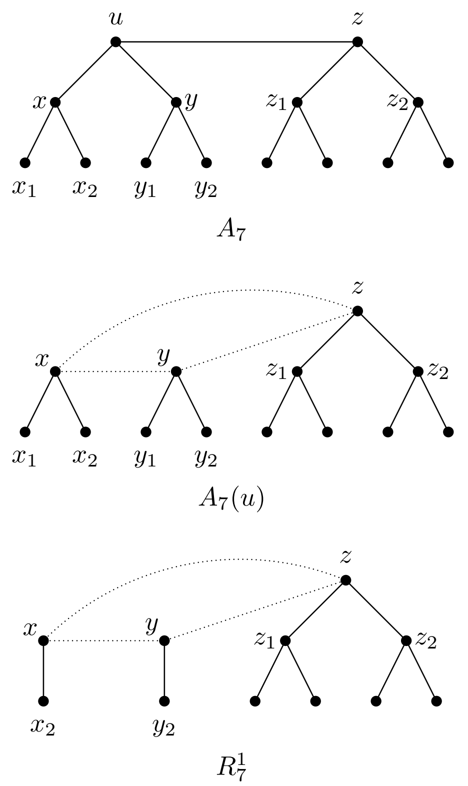


Fig. 2

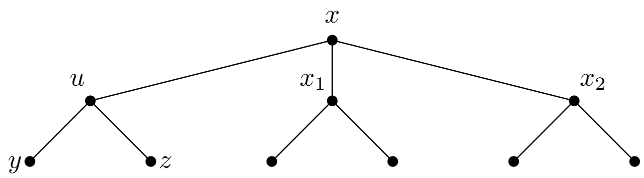
Proof. By Claims 1 and 3 the only cycle of G is a triangle C containing the vertices a and b, $C = (a, b, c, a)$, say. We shall consider two cases.

Case 1. $L_G(a) < m - 1$.
Observe that we have $L_G(a) = L_G(b)$. We shall prove that $G - \{a, b\}$ is P_m-saturated graph. Let $x, y \in V(G)$ be two nonadjacent vertices of G. Then there is a P_m through $e = xy$. Without loss of generality we may suppose $P_m = (abc \dots xy \dots)$. Then by Claim 8 there is a neighbour v of c such that $v \notin V(P_m)$.

If $d_G(v) > 1$ then we can replace the segment $\dots abc \dots$ of P_m with wvc where $w \in N_G(v) - \{c\}$ to obtain a path of order m passing through e.

If $d_G(v) = 1$ then the only path of order m through $e' = av$ must have form $P'_m = (vabc \dots)$. But then there is a path of order $m-1$ starting at a, a contradiction.

Case 2. $L_G(a) = m - 1$.
If there exists a vertex $x_2 \in N_G(c)$ such that $d_G(x_2) = 1$ and clearly $G' = (V'; E')$ with $V' = V \cup \{d, x, y\}$, $E' = E - \{ab, ac, bc\} \cup \{bd, ad, cd, xx_2, yx_2\}$ is P_m-saturated graph without cycle. Thus $n = a_m - 3$ and G is R_m^3. If non of neighbours of c is equal to 1 then by similar method as before we can show that G contains R_m^3. $\square$



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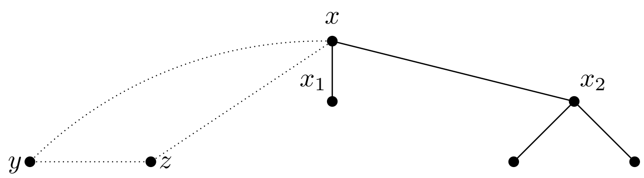

R_6^3

Fig. 3

Lemma 3. *Let G be a connected, unicyclic and P_m-saturated graph of order n without vertices of degree 2, where $6 \le m \le n < a_m$. Then G has a subgraph $H \in \{R_m^1\} \cup \mathcal{R}_m^2$.*

Proof. By Claim 1 we may suppose that $C = (c_1, c_2, c_3, c_1)$ and by Claim 7 it is sufficient to consider the following two cases.

Case 1. $L_1 = L_2 = L_3$.
Then by Claim 8 the vertices c_1, c_2, c_3 form the center of G and $L_1 = L_2 = L_3 = \frac{m}{2} - 1$ (note that m is even). Let r_1 be a neighbour of c_1 such that $L_{r_1}(c_1) = \frac{m}{2} - 1$. We shall prove that the component of $G' = (V, E - \{r_1 c_1\})$ containing the vertex r_1 contains T_H the complete binary tree with the rout r_1 with $h = \frac{m}{2} - 2$ levels. By Claim 6, r_1 has two neighbours x and y diffrent from c_1 such that $L_x(r_1) = L_y(r_1) = \frac{m}{2} - 2$. Choose one of these vertices as r_2. Having choosen r_i with $2 \le i < \frac{m}{2} - 1$ we have by Claim 6 two vertices x' and y' such that $L_{x'}(r_i) = L_{y'}(r_i) = \frac{m}{2} - (i+1)$. We may choose arbitrary one of three vertices r_{i+1} and the proof is finished in this case.

Case 2. $L_1 = L_2, L_1 < L_3$.
Let $P = (c_3 z \ldots v)$ be a path with v being a mean vertex of G. We apply the same method as in the proof of Case 1 to the graph G' obtained from G by the delition of the vertices of the path P and the cycle C with all incident edges. □

By Claims 3, 4 and 5 and Lemmas 2 and 3, we have the following

Theorem 5. *Let $6 \le m \le n < a_m$. Every connected and unicyclic P_m-saturated graph of order n contains a subgraph $H \in \mathcal{R}_m$.*

The following two lemmas are very easy to deduce.

Lemma 4. *Let u be a vertex of degree equal to two in a P_m-saturated graph G, $|G| \geq m \geq 4$. Then the neighbours of u are adjacent.*

Lemma 5. *Let $m \geq 6$. Every union of K_2 and a graph $H \in \mathcal{R}_m$ is P_m-saturated if and only if $m = 6$ and $H = R_6^1$.*

$\mathcal{H}$ is the family of graphs defined in Figure 4. Observe that for every graph $H \in \mathcal{H}$ the graphs H and $H \cup K_2$ are P_6-saturated.

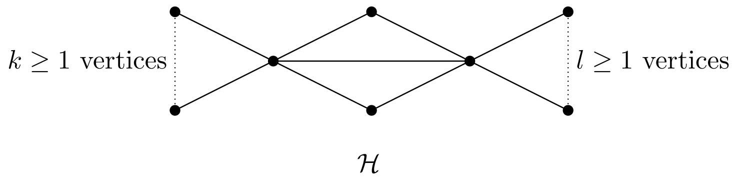

Fig. 4

Lemma 6. *Let $m \geq 6$ and let H be a connected P_m-saturated graph such that $|E(H)| = |H| + 1$. Then the graph $H \cup K_2$ is P_m-saturated if and only if $m = 6$ and $H \in \mathcal{H}$.*

Proof. Denote by u and v the vertices of K_2. Let C and C' be two different and nontrivial cycles in H and suppose that $H \cup K_2$ is P_m-saturated.

There are two possibilities: either C and C' intersect in at most one vertex or the intersection of C and C' is a non trivial path.

Thus we shall consider two cases.

Case 1. $|V(C) \cap V(C'| \leq 1$.
There is a unique path from C to C', say $P = (x_1, \ldots, x_p)$, such that $x_1 \in V(C)$, $x_p \in V(C')$ and $x_i \notin V(C) \cup V(C')$ for $i = 2, \ldots, p-1$ (possibly $p = 1$ and $V(C) \cap V(C') = \{x_1\}$). It is clear that the new edge ux_1 can not create any P_m in $H \cup K_2$.

Case 2. $|V(C) \cap V(C')| \geq 2$.
Let the path $P'' = (z_1, \ldots, z_r)$ be the intersection of the cycles C and C'. So, we have three paths, say $P = (x_1, \ldots, x_p)$, $P' = (y_1, \ldots, y_q)$ and P'' such that $x_1 = y_1 = z_1$ and $x_p = y_q = z_r$. At least two of the paths P, P' or P'' have length at least two.

The graph $H \cup K_2$ is P_m-saturated, hence there is a path of order m in the graph $H \cup K_2 \cup vx_1$, say $P''' = (u, v, x_1, \ldots)$. Since there is no P_m in the graph H we deduce easily that P''' contains all, but at most one, vertices of two paths P, P', P'' while the third of them contains at most three vertices. Without loss of generality we may suppose that either

$$P''' = (u, v, x_1, x_2, \ldots, x_p, y_{q-1}, \ldots, y_2, a_1, \ldots, a_t)$$

or

$$P''' = (u, v, x_1, x_2, \dots, x_p, y_{q-1}, \dots, y_3, a_1, \dots, a_t)$$

where $\{a_1, \dots, a_t\} \cap (V(C) \cup V(C')) = \emptyset$ and $r \le 3$. For $p \ge 3$ and $q \ge 3$ we have, by Lemma 4, $r = 2$.

Let $(b_1, \dots, b_s)$ be such a path of the graph H that $b_1 \in V(C) \cup V(C')$ and $b_2 \notin V(C) \cup V(C')$ (by consequence b_1 is the only vertex of that path which is in $V(C) \cup V(C')$). Since $H \cup vb_1$ contains P_m and there is no P_m in H, we have $s \le 2$. In particular $t \le 2$ and every vertex $x \in V(H) - (V(C) \cup V(C'))$ is pendent and adjacent to a vertex of $V(C) \cup V(C')$. Therefore joining two vertices $x, y \in V(C) \cup V(C')$ we can not obtain any path of order greater than $p + q$. Thus $m \le p + q$. Now it is very easy to deduce that $H \in \mathcal{H}$. $\square$

Theorem 6. *Let n and m be integers such that $6 \le m \le n$, $b_m \le n < a_m$ and let $G \in Sat(n; P_m)$. Then*

$$e(G) = \begin{cases} n-1 & \textit{for } \ m = 6,\ n = 8, 9 \\ n & \textit{for } \ m = 6,\ n = 6, 7 \\ n & \textit{for } \ m > 6. \end{cases}$$

Moreover, if $m > 6$ then G contains a subgraph $H \in \mathcal{R}_m$, if $m = 6$ then either G is the graph depicted in Figure 5 or $G = 2K_3$.

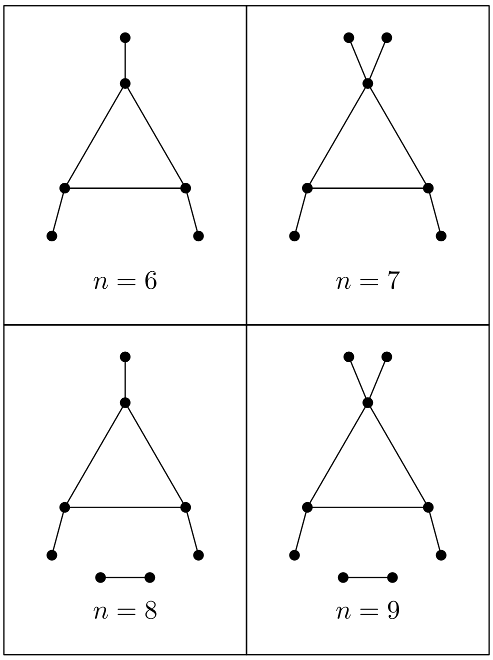

Fig. 5

Proof. Let us suppose first that one component of G is a tree. Since $n < a_m$ and by Theorem 3, the tree component of G is either K_2 or K_1. By Lemma 1 we have $sat(n, P_m) \leq n$. Thus every other component of G has at most two cycles. We conclude this case applying Theorem 5, Lemma 5 and Lemma 6.

So we may suppose that no component of G is a tree. Then the theorem follows by Theorem 5. □

Observe that since $2b_m > a_m$, for $n \geq m \geq 7$ every graph $G \in Sat(n; P_m)$ is connected.

Added in proof. We have been informed that L. W. Beineke, J. E. Dunbar and M. Frick in yet unpublished paper *Detour-saturated graphs* have given a complete characterization of unicyclic P_m-saturated graphs. We are very indebted for this information (and the preprint of the paper) to professor M. Frick. Thanks are due to professor M. Frick also for indicating an error in the previous version of our paper.

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