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**DISCRETIZATION OF THE STATIONARY
DISTRIBUTION
OF HEAT IN THE NON-HOMOGENEOUS BODY**

Abstract. We give a short survey on the theory of the mixed boundary-value problem for the stationary Fourier equation in a non-homogeneous medium defined on any Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). The compatibility condition for the thermal flux has been established by the standard procedure of integration the divergence.

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1. INTRODUCTION

The present paper supplements and continues [2, 3, 4, 5, 8] and [1] and includes a fragment of the general theory of the mixed boundary-value problem for the diffusion equation in a non-homogeneous medium. The compatibility condition (4) for the thermal flux has been established by the standard procedure of integration the divergence (see e.g. [7] chap. I, (1, sec. 1.3)).

We look for a stationary distribution of a temperature T in a solid Ω (which arises by sticking of some homogeneous materials $\Omega_1, \dots, \Omega_s$ with thermal conductivities $\theta_1, \dots, \theta_s$. Heat sources f , a temperature b on a piece $\partial_I \Omega$ of the boundary of Ω (so-called Dirichlet boundary) and overall heat-transfer coefficients α, β on the remaining part of the boundary (that is, on $\partial_{III} \Omega := (\partial \Omega) \setminus (\partial_I \Omega)$, called the Fourier boundary) are treated as known. Let us say that in any homogeneous material Ω_j the heat propagates in agreement with the Fourier law (1). On the borderland of Ω_i and Ω_j , $i \neq j$, we postulate a consistency of normal components of the heat fluxes, i.e. the local heat balance (2). Statement of this condition is a result of our

co-operation with M. Danielewski and W. Krzyżański [6]. Both of these laws have a local character (they are described by differential operators) and follow from the more fundamental integral law:

$$\frac{d}{dt} \int_B T(t, x) dx = \int_{\partial B} \mathbf{n}_B(\zeta) J(t, \zeta) d\sigma_B(\zeta) + \int_B f(t, x) dx \quad (1)$$

for every $B \subset \Omega$ with a regular boundary

where the vector-valued function J is interpreted as a heat flux, \mathbf{n}_B means the inward normal with regard to B , and σ_B – the surface element on ∂B . In the present paper we suppose that J is a “plywood” of Fickxian fluxes, i.e.

$$J(t, x) = -\theta_i \cdot \left(\frac{\partial T}{\partial x_1}(t, x), \dots, \frac{\partial T}{\partial x_n}(t, x) \right) \quad \text{for } x \in \Omega_i \quad (i = 1, \dots, s). \quad (2)$$

The law (1) is more general than appropriate differential equations, because it admits discontinuous heat fluxes; in particular the flux (2) is, in general, discontinuous.

2. STATIONARY DISTRIBUTION OF HEAT

IN A NON-HOMOGENEOUS MEDIUM – INTRODUCTION

Let $\mathbb{N} \ni n \geq 2$. Consider a bounded Lipschitz domain Ω in \mathbb{R}^n (i.e. Ω is bounded, open and connected, while the boundary $\partial\Omega$ of Ω satisfies the Lipschitz condition). Suppose that $1 \leq s \in \mathbb{N}$ and $\Omega_1, \dots, \Omega_s \in \text{top } \Omega$ (i.e. Ω_i is open in Ω , $\forall i$). Moreover $\Omega_1, \dots, \Omega_s$ are pairwise disjoint and $m\left(\Omega \setminus \bigcup_{j=1}^s \Omega_j\right) = 0$, where m stands for the Lebesgue measure in \mathbb{R}^n . We assume that for every j the boundary of Ω_j fulfills the Lipschitz condition. So, the field $\mathbf{n}^j: \partial\Omega_j \rightarrow \mathbb{R}^n$ of normal vectors, inward with respect to Ω_j , has a domain of full measure in the sense of the surface element $\sigma_j: \mathcal{B}(\partial\Omega_j) \rightarrow \mathbb{R}_+$ on $\partial\Omega_j$. The $\text{dom } \mathbf{n}^j$ may be only a subset of $\partial\Omega_j$ and the symbol $\mathcal{B}(\partial\Omega_j)$ stands for the σ -algebra of all Borel subsets of the topological space $\partial\Omega_j$. Lastly, let $\sigma: \mathcal{B}(\partial\Omega) \rightarrow \mathbb{R}_+$ denote the surface element of $\partial\Omega$. Distinguish an arbitrary Borel subset $\partial_I\Omega$ of $\partial\Omega$ and denote:

$$\partial_{III}\Omega := (\partial\Omega) \setminus (\partial_I\Omega), \quad \partial_{III}\Omega_j := (\partial\Omega_j) \cap (\partial_{III}\Omega).$$

Now we shall give the classical formulation of the mixed boundary value-problem for the stationary Fourier equation with singularities.

At this stage we do not discuss the regularity of considered functions. Assuming the following

Data:

$$\begin{aligned} \theta_1, \dots, \theta_s &\in]0, \infty[; \\ f: \Omega &\rightarrow \mathbb{R}; \end{aligned}$$

$$\begin{aligned} b &: \partial_I \Omega \rightarrow \mathbb{R}; \\ \alpha, \beta &: \partial_{III} \Omega \rightarrow \mathbb{R}, \end{aligned}$$

we look for a function

$$T: \Omega \rightarrow \mathbb{R}$$

satisfying the following

Physical laws:

$$0 = \theta_j \cdot \Delta T + f \quad \text{in } (j = 1, \dots, s) \quad (3)$$

and

$$\left(\theta_i \frac{\partial T}{\partial \mathbf{n}^i} + \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \right) (x) = 0 \quad \text{for any } i \neq j \quad (4)$$

and σ_i -almost every $x \in (\partial \Omega_i) \cap (\partial \Omega_j)$

and the

Boundary conditions:

$$T = b \quad \text{on } \partial_I \Omega, \quad (5)$$

$$\frac{\partial T}{\partial \mathbf{n}} = \alpha T + \beta \quad \text{on } \partial_{III} \Omega, \quad (6)$$

where $\frac{\partial T}{\partial \mathbf{n}^i} := \lim_{\Omega_i \ni z \rightarrow x} \mathbf{n}^i(x) \nabla_z T$ (scalar product of the vectors $\mathbf{n}^i(x)$ and $\nabla_z T$), $\nabla := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$, $\Delta := \text{div} \nabla = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\mathbf{n}: \partial \Omega \rightarrow \mathbb{R}^n$ is the inward normal with respect to Ω .

To put a definition of a solution of the problem (3)–(6) with full mathematical accuracy, we have to go across the following

Heuresis. Let $\varphi: \bar{\Omega} \rightarrow \mathbb{R}$ vanish on $\partial_I \Omega$. Using the Fourier equation in Ω_j , integrating by parts and applying (6) we calculate:

$$\begin{aligned} 0 &= \int_{\Omega_j} (\theta_j \Delta T + f) \varphi \, dm = \\ &= \int_{\Omega_j} f \varphi \, dm - \theta_j \int_{\partial \Omega_j} \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j - \theta_j \int_{\Omega_j} (\nabla T)(\nabla \varphi) \, dm = \\ &= \int_{\Omega_j} f \varphi \, dm - \int_{\Omega \cap \partial \Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j - \theta_j \int_{\partial_{III} \Omega_j} (\alpha T + \beta) \varphi \, d\sigma - \theta_j \int_{\Omega_j} (\nabla T)(\nabla \varphi) \, dm. \end{aligned}$$

Adding the obtained equalities within the range of $1 \leq j \leq s$ we get:

$$\begin{aligned} \sum_{j=1}^s \theta_j \left(\int_{\Omega_j} (\nabla T)(\nabla \varphi) \, dm + \int_{\partial_{III} \Omega_j} (\alpha T + \beta) \varphi \, d\sigma \right) + \sum_{j=1}^s \int_{\Omega \cap \partial \Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = \\ = \int_{\Omega} f \varphi \, dm. \end{aligned} \quad (7)$$

We denote

$$M_{ij} := \Omega \cap (\partial\Omega_i) \cap (\partial\Omega_j) \quad \text{for different } i, j \in \{1, \dots, s\}$$

and remark that

$$\Omega \cap \partial\Omega_j = \bigcup_{i \neq j} M_{ij}, \quad \sigma_j(M_{i_1, j} \cap M_{i_2, j}) = 0 \quad \text{for } i_1 \neq i_2.$$

Hence, using (4) and dividing the set $\mathcal{A} := \{(i, j) \in \{1, \dots, s\}^2: i \neq j\}$ into packets, we calculate:

$$\begin{aligned} \sum_j \int_{\Omega \cap \partial\Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j &= \sum_{(i, j) \in \mathcal{A}_{M_{ij}}} \int \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = \sum_{(i, j) \in \mathcal{A}_{M_{ji}}} \int \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_i = \\ &= \sum_{(k, l) \in \mathcal{A}_{M_{kl}}} \int \theta_k \frac{\partial T}{\partial \mathbf{n}^k} \varphi \, d\sigma_l = - \sum_{(k, l) \in \mathcal{A}_{M_{kj}}} \int \theta_l \frac{\partial T}{\partial \mathbf{n}^l} \varphi \, d\sigma_l = - \sum_{(i, j) \in \mathcal{A}_{M_{ij}}} \int \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j. \end{aligned}$$

Thus $\sum_j \int_{\Omega \cap \partial\Omega_j} \theta_j \frac{\partial T}{\partial \mathbf{n}^j} \varphi \, d\sigma_j = 0$, which reduces the equality (7) to the form:

$$a_{\theta, \alpha}(T, \varphi) = \xi(\varphi), \quad (8)$$

where

$$a_{\theta, \alpha}(T, \varphi) := \sum_{j=1}^s \theta_j \left(\int_{\Omega_j} (\nabla T)(\nabla \varphi) \, dm + \int_{\partial_{III} \Omega_j} \alpha T \varphi \, d\sigma \right), \quad (9)$$

$$\xi(\varphi) := \int_{\Omega} f \varphi \, dm - \sum_{j=1}^s \theta_j \int_{\partial_{III} \Omega_j} \beta \varphi \, d\sigma \quad (10)$$

and $\theta := (\theta_1, \dots, \theta_s) \in \mathbb{R}^s$. \square

Now we shall present an exact formulation of the boundary-value problem (3)–(6). In order to attain this we introduce the following special space of test functions:

$$\Phi := \{\varphi \in H^1: \varphi|_{\partial_I \Omega} = 0\} \quad (11)$$

where as usually

$$H^1(\Omega) := \left\{ \varphi \in L^2(\Omega): \frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n} \in L^2(\Omega) \right\}$$

and $\frac{\partial \varphi}{\partial x_i}$ is the only element of $L^2(\Omega)$ for which

$$\left(\frac{\partial \varphi}{\partial x_i} \mid \psi \right)_{L^2(\Omega)} = - \left(\varphi \mid \frac{\partial \psi}{\partial x_i} \right)_{L^2(\Omega)} \quad (12)$$

Here

$$\mathfrak{D}(\Omega) := C^\infty(\Omega) \cap \{\text{compact support}\} = \left\{ \psi: \Omega \rightarrow \mathbb{R} \text{ of class } C^\infty \mid \overline{\{\psi \neq 0\}} \subset \Omega \right\},$$

while $L^2(\Omega)$ is defined as the factor space:

$$L^2(\Omega) := \{\text{square summable real functions on } \Omega\} / \{\text{almost everywhere vanishing functions on } \Omega\}.$$

The linear spaces $L^2(\Omega)$, $H^1(\Omega)$ are separable Hilbert spaces with the scalar products

$$\begin{aligned} (p_1 \mid p_2)_{L^2(\Omega)} &:= \int_{\Omega} p_1 p_2 \\ (\varphi_1 \mid \varphi_2)_{H^1(\Omega)} &:= (\varphi_1 \mid \varphi_2)_{L^2(\Omega)} + (\nabla \varphi_1 \mid \nabla \varphi_2)_{L^2(\Omega, \mathbb{R}^n)} = \\ &= (\varphi_1 \mid \varphi_2)_{L^2(\Omega)} + \sum_{i=1}^n \left(\frac{\partial \varphi_1}{\partial x_i} \mid \frac{\partial \varphi_2}{\partial x_i} \right)_{L^2(\Omega)} \end{aligned} \quad (13)$$

respectively.

The restriction $\varphi|_{\partial_I \Omega}$ on the right hand-side of (11) is meant in the sense of the trace theory (see [9, 10]), i.e. it is a limit of the sequence $\left((\varphi_\nu)|_{\partial_I \Omega} \right)_{\nu=1}^\infty$ in the Hilbert space $L^2(\partial_I \Omega)$ (provided with the scalar product $(b_1 \mid b_2)_{L^2(\partial_I \Omega)} := \int_{\partial_I \Omega} b_1 b_2 d\sigma$) for any sequence $(\varphi_\nu)_{\nu=1}^\infty \in (C^\infty(\mathbb{R}^n))^{\mathbb{N}}$ such that $(\varphi_\nu)|_{\Omega} \xrightarrow{\nu \rightarrow \infty} \varphi$ in $H^1(\Omega)$ (see also (26) below). So, Φ is a separable Hilbert space with the inner product (13), as a closed subspace of $H^1(\Omega)$.

Definition 1. A function $T \in H^1(\Omega)$ is a weak solution of the problem (3)–(6) for given:

- a) $\theta_1, \dots, \theta_s \in]0, \infty[;$
- b) $0 \leq \alpha \in \begin{cases} L^{n-1}(\partial_{III} \Omega), & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III} \Omega) \text{ for some } \varepsilon > 0, & \text{as } n = 2; \end{cases}$
- c) $\xi \in \Phi'$;
- d) $b = \bar{b}|_{\partial_I \Omega}$ for some $\bar{b} \in H^1(\Omega)$

if and only if $T|_{\partial_I \Omega} = b$ and (8) holds for any $\varphi \in \Phi$.

All the restrictions to $\partial_I \Omega$ or $\partial_{III} \Omega$ are here meant in the sense of the trace theory. In particular, the restriction $(\alpha T \varphi)|_{\partial_{III} \Omega}$ on the right hand-side of (9) is σ -summable, in virtue of (26). In Definition 1 we did not specify regularity of the

data f , α , β considered in the classical statement of (3)–(6); we only assume that the linear functional (10) is well defined and continuous as a function on the Hilbert space Φ , or, in other words, ξ is a element of the dual space Φ' . The condition $T|_{\partial_I\Omega} = b$ and the variational equation (8) take in both the physical laws (3), (4) and the boundary conditions (5), (6).

Example 3. Consider

$$f \in \begin{cases} L^{\frac{2n}{n+2}}(\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\Omega) & \text{as } n = 2 \end{cases}$$

and

$$\beta \in \begin{cases} L^{2-\frac{2}{n}}(\partial_{III}\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III}\Omega) & \text{as } n = 2 \end{cases} \quad (14)$$

for a certain $\varepsilon > 0$. Then the functional $\xi: \Phi \rightarrow \mathbb{R}$ given by the formula (1) is well-defined and continuous. It follows from (26).

Physically, the functional ξ from Example 3 corresponds to *voluminal* heat sources f (that means that the second integral on the right hand-side of (1) is arbitrarily small for sufficiently small $m(B)$). Definition 1 admits also singular heat sources, e.g.

$$\xi: \Phi \ni \varphi \rightarrow \int_{\mathcal{M}} \rho \varphi d\sigma_{\mathcal{M}} - \sum_{j=1}^s \theta_j \int_{\partial_{III}\Omega_j} \beta \varphi d\sigma \in \mathbb{R} \quad (15)$$

where β satisfies (14), while $\mathcal{M} \subset \overline{\Omega}$ is a hypersurface in \mathbb{R}^n (i.e. \mathcal{M} is a submanifold of \mathbb{R}^n , $\text{codim } \mathcal{M} = 1$), $\sigma_{\mathcal{M}}$ is the surface element on \mathcal{M} and, finally,

$$\rho \in \begin{cases} L^{2-\frac{2}{n}}(\partial_{III}\Omega) & \text{as } n > 2, \\ L^{1+\varepsilon}(\partial_{III}\Omega) & \text{as } n = 2. \end{cases} \quad (16)$$

The functional (15) belongs to Φ' because the condition (16) has the same structure as (14). Physically, the function $\rho: \mathcal{M} \rightarrow \mathbb{R}$ is a density of the *surface* heat sources.

Proposition 1. The following conditions are equivalent:

- a) $a_{\theta, \alpha}$ is an admissible scalar product in Φ (i.e. it induces in Φ the topology inherited from $H^1(\Omega)$);
- b) $\sigma(\{\alpha \neq 0\} \cup \partial_I\Omega) > 0$.

Proof. (a) \Rightarrow (b). Suppose, contrary to our claim, that

$$\sigma(\{\alpha \neq 0\} \cup \partial_I\Omega) = 0. \quad (17)$$

Then $\Phi = H^1(\Omega)$, while $\varphi \equiv 1$ is an element of Φ , so,

$$0 < a_{\theta,\alpha}(\varphi, \varphi) = \sum_{j=1}^s \theta_j \left(\int_{\Omega_j} (\nabla\varphi)(\nabla\varphi) dm + \underbrace{\int_{\partial_{III}\Omega_j} \alpha\varphi\varphi d\sigma}_0 \right) = 0.$$

In this way the hypothesis (17) has led to a contradiction.

(b) \Rightarrow (a). The bilinear form $a_{\theta,\alpha}: \Phi \times \Phi \rightarrow \mathbb{R}$ is continuous, symmetric and non-negative, so it induces the semi-norm:

$$\Phi \ni \varphi \rightarrow |\varphi|_{\theta,\alpha} := \sqrt{a_{\theta,\alpha}(\varphi, \varphi)} \in \mathbb{R}_+. \quad (18)$$

It is sufficient to show that the semi-norm (18) is stronger than the original norm

$$\|\varphi\|_{H^1(\Omega)} := \sqrt{(\varphi | \varphi)_{H^1(\Omega)}} \quad (19)$$

induced by the inner product (13). Of course, for any $\varphi \in \Phi$

$$\begin{aligned} |\varphi|_{\theta,\alpha}^2 &= \sum_{j=1}^s \theta_j \left(\int_{\Omega_j} |\nabla\varphi|^2 dm + \int_{\partial_{III}\Omega_j} \alpha\varphi^2 d\sigma \right) \geq \\ &\geq \left(\min_i \theta_i \right) \left(\|\nabla\varphi\|_{L^2} + \int_{\partial_{III}\Omega} \alpha\varphi^2 d\sigma \right). \end{aligned} \quad (20)$$

Consider an arbitrary Borel set $\Sigma \subset \partial\Omega$ for which $\sigma(\Sigma) > 0$. Then by Lemma 1 which will be proved below the functional

$$H^1(\Omega) \ni \varphi \rightarrow |\varphi|_{\Sigma} := \sqrt{\|\nabla\varphi\|_{L^2}^2 + \frac{1}{\sigma(\Sigma)} \left(\int_{\Sigma} \varphi d\sigma \right)^2} \in \mathbb{R}_+ \quad (21)$$

is an admissible norm in $H^1(\Omega)$, so, $\exists c_{\Sigma} \in \mathbb{R}_+ \forall \varphi \in H^1(\Omega): |\varphi|_{\Sigma} \geq c_{\Sigma} \|\varphi\|_{H^1(\Omega)}$.

In the case of $\sigma(\partial_I\Omega) > 0$ we put $\Sigma = \partial_I\Omega$ and, remembering (20), obtain:

$$\|\varphi\|_{\theta,\alpha} \geq \left(\min_i \theta_i \right) \|\nabla\varphi\|_{L^2} = \left(\min_i \theta_i \right) |\varphi|_{\Sigma} \geq \left(\min_i \theta_i \right) \cdot c_{\Sigma} \cdot \|\varphi\|_{H^1(\Omega)}$$

for any $\varphi \in H^1(\Omega)$. q.e.d.

It remains the case $\sigma(\partial_I\Omega) = 0$. Then

$$0 < \sigma(\{\alpha \neq 0\}) = \sigma \left(\bigcup_{\mathbb{Q} \ni \varepsilon > 0} \{\alpha > \varepsilon\} \right) \leq \sum_{\mathbb{Q} \ni \varepsilon > 0} \sigma(\{\alpha > \varepsilon\})$$

and so, there is an $\varepsilon_* \in]0, 1[$ such that $\sigma(\{\alpha > \varepsilon_*\}) > 0$. Putting $\Sigma = \{\alpha > \varepsilon_*\}$ and using (20) we estimate:

$$\begin{aligned}
|\varphi|_{\theta, \alpha}^2 &\geq \left(\min_i \theta_i\right) \left(\|\nabla \varphi\|_{L^2}^2 + \varepsilon_* \cdot \int_{\{\alpha > \varepsilon_*\}} \varphi^2 d\sigma \right) \geq \\
&\geq \left(\min_i \theta_i\right) \cdot \varepsilon_* \cdot \left(\|\nabla \varphi\|_{L^2}^2 + \int_{\{\alpha > \varepsilon_*\}} \varphi^2 d\sigma \right) \geq \\
&\geq \varepsilon_* \cdot \left(\min_i \theta_i\right) \cdot \left(\|\nabla \varphi\|_{L^2}^2 + \frac{1}{\sigma(\Sigma)} \int_{\Sigma} \varphi^2 d\sigma \right) = \\
&= \varepsilon_* \cdot \left(\min_i \theta_i\right) \cdot |\varphi|_{\Sigma}^2 \geq \varepsilon_* \cdot \left(\min_i \theta_i\right) \cdot c_{\Sigma}^2 \cdot \|\varphi\|_{H^1(\Omega)}^2. \quad \square
\end{aligned}$$

Lemma 1. *Consider $\Sigma \in \{\sigma > 0\}$. Then the functional (21) is an admissible norm in the Sobolev space $H^1(\Omega)$.*

Proof. Of course the semi-norm (21) is $H^1(\Omega)$ -continuous. In other words, the norm (19) is stronger than the semi-norm (21).

Suppose $|\varphi|_{\Sigma} = 0$ for a fixed $\varphi \in H^1(\Omega)$. Then $\|\nabla \varphi\|_{L^2} = \int_{\Sigma} \varphi d\sigma = 0$. But Ω is connected, so, be the generalized du Bois-Reymond Lemma, $\varphi \equiv c$ for some $c \in \mathbb{R}$. Consequently,

$$0 = \int_{\Sigma} \varphi d\sigma = c \cdot \sigma(\Sigma)$$

and $c = 0$, $\varphi \equiv 0$. Thus, $|\cdot|_{\Sigma}$ is a norm.

In those circumstances, it is sufficient to show (thanks to the Banach open mapping theorem) that the norm (21) is complete, i.e. the pair $(H^1(\Omega), |\cdot|_{\Sigma})$ is a Banach space. And so, assume that a sequence $(\varphi_{\nu}) \in (H^1(\Omega))^{\mathbb{N}}$ is regular in the sense of $|\cdot|_{\Sigma}$. Then the sequences $(\nabla \varphi_{\nu})$ and $(\int_{\Sigma} \varphi_{\nu} d\sigma)$ are regular in the Hilbert spaces $L^2 := L^2(\Omega, \mathbb{R}^n)$ (of square summable vector fields) and \mathbb{R} , respectively. Therefore, there are $F \in L^2$ and $r \in \mathbb{R}$, such that

$$\lim_{\nu \rightarrow \infty} \|\nabla \varphi_{\nu} - F\|_{L^2} = 0 \quad \text{and} \quad \lim_{\nu \rightarrow \infty} \int_{\Sigma} \varphi_{\nu} d\sigma = r.$$

In particular, by the definition (12), for any solenoidal test vector field $\psi: \Omega \rightarrow \mathbb{R}^n$ (i.e. $\psi_1, \dots, \psi_n \in \mathfrak{D}(\Omega)$, $\operatorname{div} \psi = \sum_{i=1}^n \frac{\partial \psi_i}{\partial x_i} \equiv 0$) we have:

$$(F | \psi)_{L^2} \stackrel{\infty \leftarrow \nu}{=} (\nabla \varphi_{\nu} | \psi)_{L^2} = -(\varphi_{\nu} | \operatorname{div} \psi)_{L^2(\Omega)} = 0.$$

In view of the arbitrariness of ψ , the regular distribution

$$[F]: \mathfrak{D}(\Omega) \ni \lambda \xrightarrow{df} \int_{\Omega} \lambda(x) F(x) dx \in \mathbb{R}^n$$

is *conservative*, i.e. there exists a scalar distribution $V \in \mathfrak{D}'(\Omega)$ for which

$$[F_i] = -\frac{\partial V}{\partial x_i} \quad \text{for any } 1 \leq i \leq n. \quad (22)$$

It follows from the de Rham theorem (see [12]). Simultaneously, $\partial\Omega$ fulfils the Lipschitz condition, thus, V is a regular “square – summable” distribution, i.e. there is a function $v \in L^2(\Omega)$ such that $V = [v]$ (see [13, 11], or [10]). By (22), $v \in H^1(\Omega)$ and $\forall 1 \leq i \leq n \frac{\partial v}{\partial x_i} = F_i$ in the sense of the definition (12). Finally, the function

$$\underbrace{v}_{\in H^1(\Omega)} + \underbrace{\frac{1}{\sigma(\Sigma)} \left(r - \int_{\Sigma} v d\sigma \right)}_{\text{constant function}} \quad (\in H^1(\Omega))$$

is the limit of (φ_ν) in the normed space $(H^1(\Omega), |\cdot|_{\Sigma})$. Then, the norm (21) is complete. \square

In the same way we can prove the following analogons

Lemma 2. *The functional*

$$H^1(\Omega) \ni \varphi \rightarrow |\varphi|_{\mathcal{P}} := \sqrt{\int_{\Omega} |\nabla_x \varphi|^2 dx + \left(\int_{\Omega} \varphi(x) dx \right)^2} \in \mathbb{R}_+$$

is an admissible norm in $H^1(\Omega)$.

In particular, there is a constant $C \in \mathbb{R}_+$, dependent only on Ω , such that

$$\|\varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{\mathcal{P}} \quad \text{for any } \varphi \in H^1(\Omega). \quad (23)$$

It is one of two well-known Poincaré inequalities.

We say that a sequence $(E_N)_{N \in \mathbb{N}}$ of finite-dimensional linear subspaces of a normable space E is an *inner approximation* of E if for every point $e \in E$

$$\lim_{N \rightarrow \infty} \text{dist}(e, E_N) = 0, \quad (24)$$

where $\text{dist}(e, E_N) := \inf \{|e - \varphi| : \varphi \in E_N\}$ and $|\cdot|$ is an admissible norm in E (of course, the condition (24) does not depend on choice of such a norm). Fixing a norm $|\cdot|_E$ in E we may consider the Banach space $M_2(E)$ of all bilinear continuous forms on $E \times E$, with the norm

$$\|\gamma\|_{M_2(E)} := \sup \{|\gamma(e_1, e_2)| : |e_1|_E \leq 1, |e_2|_E \leq 1\}.$$

Now, we will prove a theorem on the existence, uniqueness and continuous dependence of the solution of (3)–(6) on the data. The solution T will be constructed as a limit of the sequence $(T_N)_{N=0}^{\infty}$ in $H^1(\Omega)$.

Theorem 1. *Suppose $\sigma(\{\alpha \neq 0\} \cup \partial_I \Omega) > 0$. Then, for arbitrary given data a)–d) (from Definition 1), there exists the only solution T of the problem (3)–(6). Moreover, if:*

$$\begin{aligned} \bar{b} &\in H^1(\Omega) \text{ is any extension of } b \text{ (i.e. } b = \bar{b}|_{\partial_I \Omega}), \\ (\Phi_N)_{N \in \mathbb{N}} &\text{ is an inner approximation of } \Phi, \end{aligned}$$

and:

$$\begin{aligned} \theta &\xrightarrow{N \rightarrow \infty} \theta \text{ in } \mathbb{R}^s, \\ \alpha_N &\xrightarrow{N \rightarrow \infty} \alpha \text{ in } \begin{cases} L^{n-1}(\partial_{III} \Omega), & \text{when } n > 2 \\ L^{1+\varepsilon}(\partial_{III} \Omega), & \text{when } n = 2, \end{cases} \\ \bar{b}_N &\xrightarrow{N \rightarrow \infty} \bar{b} \text{ in } H^1(\Omega), \\ \xi_N &\xrightarrow{N \rightarrow \infty} \xi \text{ in } \Phi', \end{aligned}$$

then:

$$\text{a) } \forall! N \in \mathbb{N} \quad \exists! u_N \in \Phi_N:$$

$$a_{\theta, \alpha_N}(u_N, \varphi) = \xi_N(\varphi) - a_{\theta, \alpha_N}(\bar{b}_N, \varphi) \quad \text{for } \varphi \in \Phi_N; \quad (25)$$

$$\text{b) } T_N := u_N + \bar{b}_N \xrightarrow{N \rightarrow \infty} T \text{ in } H^1(\Omega).$$

The symbols "∀!" and "∃!" above denote "for sufficiently large" and "there exists the unique", respectively (see, at once the comment (27)).

Proof. Existence. In agreement with Proposition 1, the bilinear form $a_{\theta, \alpha}$ (in restriction to $\Phi \times \Phi$) is an admissible scalar product in Φ . Let $u \in \Phi$ mean the Riesz representation of the functional

$$l: \Phi \ni \varphi \xrightarrow{\text{df}} \xi(\varphi) - a_{\theta, \alpha}(\bar{b}, \varphi) \in \mathbb{R}$$

in the sense of the scalar product $a_{\theta, \alpha}$. Then

$$T := u + \bar{b}$$

is a solution of the problem (3)–(6) because $T|_{\partial_I \Omega} = \bar{b}|_{\partial_I \Omega} = b$ and for any $\varphi \in \Phi$:

$$a_{\theta, \alpha}(T, \varphi) = \underbrace{a_{\theta, \alpha}(u, \varphi)}_{l(\varphi)} + a_{\theta, \alpha}(\bar{b}, \varphi) = \xi(\varphi).$$

Uniqueness. Let \tilde{T} be a solution of (3)–(6). Then $T - \tilde{T} \in \Phi$ and for each $\varphi \in \Phi$:

$$a_{\theta, \alpha}(T - \tilde{T}, \varphi) = a_{\theta, \alpha}(T, \varphi) - a_{\theta, \alpha}(\tilde{T}, \varphi) = \xi(\varphi) - \xi(\varphi) = 0.$$

Putting $\varphi = T - \tilde{T}$ we get: $T - \tilde{T} = 0$.

Approximation. For a fixed exponent $r \in [1, \infty[$ consider the Sobolev space

$$W^{1,r}(\Omega) := \left\{ \varphi \in L^r(\Omega) : \underbrace{\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n}} \in L^r(\Omega) \right\}.$$

in the sense of definition (12)

It is a Banach space with the norm

$$\|\varphi\|_{W^{1,r}(\Omega)} := \left(\int_{\Omega} |\varphi(x)|^r dx + \sum_{i=1}^n \int_{\Omega} \left| \frac{\partial \varphi}{\partial x_i}(x) \right|^r dx \right)^{\frac{1}{r}}.$$

Using fundamental facts from the trace theory and the Sobolev embedding theorems (see [10, 9]) one can prove that the restriction operator

$$W^{1,r}(\Omega) \xrightarrow{\text{restr.}} L^{\frac{(n-1)r}{n-r}}(\partial\Omega) \quad (26)$$

is well defined and continuous.

Hence, in particular,

$$a_{\theta, \alpha_N} \xrightarrow{N \rightarrow \infty} a_{\theta, \alpha} \quad \text{in } M_2(H^1(\Omega)).$$

Consequently, the sequence $(l_N)_{N \in \mathbb{N}}$ of functionals

$$l_N : \Phi \ni \varphi \xrightarrow{df} \xi_N(\varphi) - a_{\theta, \alpha_N}(\bar{b}_N, \varphi) \in \mathbb{R}$$

converges to l in the Banach space Φ' , as $N \rightarrow \infty$. By Lemma 3,

$$u_N \xrightarrow{N \rightarrow \infty} u \quad \text{in } \Phi.$$

Finally,

$$T_N = u_N + \bar{b}_N \xrightarrow{N \rightarrow \infty} u + \bar{b} = T \quad \text{in } H^1(\Omega). \quad \square$$

The after-mentioned assertion, originating mainly from Ritz and Riesz, is a result of great weight in the discretization problems. Namely, modifying slightly the proof of so-called *projective theorem* (see [14]) we obtain the following

Lemma 3. *Let $(E_N)_{N \in \mathbb{N}}$ be an inner approximation of a Hilbert space E . Consider a functional $l \in E'$ and a bilinear continuous form $a : E \times E \rightarrow \mathbb{R}$ satisfying the condition*

$$\inf_{\varphi \in E \setminus \{0\}} \frac{a(\varphi, \varphi)}{|\varphi|_E^2} > 0$$

(where $|\cdot|_E$ is the norm of E). Consider additionally sequences

$$(l_N) \in (E')^{\mathbb{N}} \quad \text{and} \quad (a_N \in M_2(E)^{\mathbb{N}})$$

convergent to l and a , respectively (in the Banach spaces E' and $M_2(E)$, respectively).

Then:

a) $\forall! N \in \mathbb{N} \quad \exists! u_N \in E_N$:

$$a_N(u_N, \varphi) = l_N(\varphi) \quad \text{for any } \varphi \in E_N;$$

b) $u_N \xrightarrow{N \rightarrow \infty} u$ in E ,

where u is the only vector of E such that $a(u, \cdot) = l$.

Certainly, if a is additionally symmetric, then a is an admissible scalar product in E , while the vector u is the Riesz representation of l in the Hilbert space (E, a) . Remark that there exists such large integer $N_0 \in \mathbb{N}$ as the form a_N is coercitive for every $N \geq N_0$. Then the part a) of the above proposition takes the form “ $\forall N \geq \mathbb{N} \exists! u_N \in E_N \dots$ ”. This concerns also Theorem 1. In particular,

$$\text{if } \theta \in]0, \infty[^s \text{ and } \alpha_N \geq 0 \text{ for every } N \in \mathbb{N}, \text{ then we can put } N_0 = 0 \quad (27)$$

(i.e. the part a) of proposition of Theorem 1 takes the form “ $\forall N \in \mathbb{N} \exists! u_N \in \Phi \dots$ ”. The assumption “ $\forall N \in \mathbb{N}: \theta \in]0, \infty[^s$ ” is natural whereas the approximation α_N do not have to be non-negative for some first N . For example, it may happen when α_N is the finite expansion of α in a Fourier series.

The pure Neuman boundary value problem needs an individual treatment, because in the case of $\alpha = 0$ and $\partial_I \Omega = \emptyset$ we can not use Proposition 1.

Theorem 2. Assume $\alpha = 0$ and $\partial_I \Omega = \emptyset$. Consider arbitrary data $\theta = (\theta_1, \dots, \theta_s) \in]0, \infty[^s$ and $\xi \in (H^1(\Omega))'$ (see a), c) from Definition 1).

1) Then the following conditions are equivalent:

- (i) there exists a solution of the problem (3)–(6);
- (ii) $\xi(\mathbf{1}) = 0$, where $\mathbf{1}(\in H^1(\Omega))$ is the function equal identically to 1 ($\in \mathbb{R}$).

2) Suppose that $\xi(\mathbf{1}) = 0$ and consider an arbitrary number $h \in \mathbb{R}$. Then there exists the unique solution T of (3)–(6) such that

$$\int_{\Omega} T(x) dx = h. \quad (28)$$

Furthermore, if $(E_N)_{N \in \mathbb{N}}$ is an inner approximation of the Hilbert space $H^1(\Omega) \cap \{ \int = 0 \}$ and:

$$h_N \xrightarrow{N \rightarrow \infty} h \quad \text{in } \mathbb{R},$$

$$\theta_N \xrightarrow{N \rightarrow \infty} \theta \quad \text{in }]0, \infty[^s,$$

$$\xi_N \xrightarrow{N \rightarrow \infty} \xi \quad \text{in } (H^1(\Omega))',$$

then:

a) $\forall N \in \mathbb{N} \exists! u_N \in E_N$:

$$a_{\theta,0}(u_N, \varphi) = \xi_N(\varphi) \quad \text{for } \varphi \in E_N; \quad (29)$$

b) $T_N := u_N + \frac{1}{m(\Omega)} h_N \xrightarrow{N \rightarrow \infty} T$ in $H^1(\Omega)$.

The linear subspace

$$H^{1,0}(\Omega) := \left\{ \varphi \in H^1(\Omega) : \int_{\Omega} \varphi(x) dx = 0 \right\}$$

of $H^1(\Omega)$ is closed in the Sobolev space $H^1(\Omega)$, so it is a Hilbert space with the inner product (13) (in fact, it is a closed hyperplane in $H^1(\Omega)$).

Proof. Since for any $\varphi \in H^1(\Omega)$

$$a_{\theta,0}(\varphi, \varphi) = \sum_{j=1}^s \theta_j \int_{\Omega} |\nabla_x \varphi|^2 dx \geq \left(\min_{1 \leq i \leq s} \theta_i \right) \int_{\Omega} |\nabla_x \varphi|^2 dx, \quad (30)$$

the restriction of the bilinear form $a_{\theta,0}$ to $(H^{1,0}(\Omega))^2$ is a scalar product inducing the norm (18) which, on the strength of Lemma 2, is equivalent to the original norm (19).

Suppose T is a solution of (3)–(6) fulfilling the condition (28). Then the centring-function

$$u := T - \frac{1}{m(\Omega)} h$$

is in the hyperplane $H^{1,0}(\Omega)$ and for every $\varphi \in H^1(\Omega)$:

$$a_{\theta,0}(u, \varphi) = a_{\theta,0} \left(u + \frac{1}{m(\Omega)} h, \varphi \right) = a_{\theta,0}(T, \varphi) = \xi(\varphi) \quad (31)$$

because of the obvious equality

$$a_{\theta,0}(1, \varphi) = 0. \quad (32)$$

Ad 1) (i) \Rightarrow (ii). We suppose there is a solution T of (3)–(6). In accordance with (31),

$$\xi(\mathbf{1}) = a_{\theta,0}(T, \mathbf{1}) = 0.$$

(ii) \Rightarrow (i). Let $u \in H^{1,0}(\Omega)$ stand for the Riesz representation of the functional $\xi|_{H^{1,0}(\Omega)}$ in the sense of scalar product $a_{\theta,0}|_{(H^{1,0}(\Omega))^2}$. Then the function

$$T := u + \frac{1}{m(\Omega)} h \quad (33)$$

is a solution of (3)–(6) satisfying (28). Indeed, remembering (32), we have:

$$\begin{aligned} a_{\theta,0}(T, \varphi) &= a_{\theta,0}(u, \varphi) = a_{\theta,0}\left(u, \underbrace{\varphi - \frac{1}{m(\Omega)} \int_{\Omega} \varphi dm}_{\in H^{1,0}(\Omega)}\right) = \\ &= \xi\left(\varphi - \frac{1}{m(\Omega)} \int_{\Omega} \varphi dm\right) = \xi(\varphi) \end{aligned}$$

for every $\varphi \in H^1(\Omega)$.

Ad 2) Assume $\xi(\mathbf{1}) = 0$ and consider $h \in \mathbb{R}$. In the previous step we have showed that the function (33) is a solution of (3)–(6) satisfying (28).

Uniqueness. Let \tilde{T} be a solution of the problem (3)–(6), (28). Then the difference $T - \tilde{T}$ belongs to $H^{1,0}(\Omega)$ and

$$a_{\theta,0}(T - \tilde{T}, \varphi) = a_{\theta,0}(T, \varphi) - a_{\theta,0}(\tilde{T}, \varphi) = \xi(\varphi) - \xi(\varphi) = 0$$

for each $\varphi \in H^1(\Omega)$. Putting $\varphi = T - \tilde{T}$ we get: $T - \tilde{T} = 0$.

Approximation. By Lemma 3, there holds part a) of the proposition and $u_N \xrightarrow{N \rightarrow \infty} u$ in the Hilbert space $H^{1,0}(\Omega)$. Finally,

$$T_N = u_N + \frac{1}{m(\Omega)} h_N \xrightarrow{N \rightarrow \infty} u + \frac{1}{m(\Omega)} h = T \quad \text{in } H^1(\Omega). \quad \square$$

In the heat theory the above number h is interpreted as the *total heat* of the medium.

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