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A MAJORIZATION RELATION FOR A CERTAIN CLASS OF $*$-QUIVERS WITH AN ORTHOGONALITY CONDITION

Abstract. In [1, 2, 3], $*$-algebras and $*$-categories over the field $\mathbb{C}$ of complex numbers were quasi-ordered with respect to the complexity of the structure of their $*$-representations with a majorization relation $\succ$. A notion of $*$-wildness was also introduced there for an algebra (a category) if the algebra majorizes the $*$-algebra $C^*(\mathcal{F}_2)$. In this paper, we discuss some methods for proving that an algebra is $*$-wild and obtain criteria for certain "standard" $*$-categories (ensembles with an orthogonality condition) to be $*$-wild.

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1. INTRODUCTION

A majorization relation $\succ$ for $*$-algebras and $*$-categories was introduces in [1, 2, 3] as follows: for categories $\mathcal{K}_1$ and $\mathcal{K}_2$, the majorization $\mathcal{K}_1 \succ \mathcal{K}_2$ means that the problem of classifying $*$-representations of the category $\mathcal{K}_2$ over the category of Hilbert spaces is contained in that for the category $\mathcal{K}_1$.

Let Q be a $*$-quiver with relations. Let it generate a $*$-category $\mathcal{K} = \mathcal{K}(Q)$ over the field $\mathbb{C}$ of complex numbers, see [4, 5]. Let $\operatorname{Rep}\mathcal{K}$ ($\operatorname{Rep} Q$) be the category of $*$-representations of the $*$-category $\mathcal{K}$ ($*$-quiver Q), see Section 2 for the definitions.

Let $\mathcal{F}_2 = \mathbb{C} < u_1, u_2 \mid u_i u_i^* = u_i^* u_i = e,\ i = 1, 2 >$. A $*$-category $\mathcal{K}$ is said to be $*$-wild [1, 2, 3] if $\mathcal{K} \succ \mathcal{F}_2$. The wildness indicates that the classification problem is extremely complex; in a certain sense, $\mathcal{F}_2$ majorizes "all other" $*$-algebras and $*$-categories, see [6].

An involution quiver Q will be called an *ensemble* if the sets of its vertices and edges can be written as $Q_v = Q_v^{(1)} \cup Q_v^{(2)}$ and $Q_a = Q_a^{(1)} \cup Q_a^{(2)}$, respectively. Here the subsets that enter in the unions can have a nonempty intersection and even coincide. They are defined as follows. For every pair of vertices in the sets $Q_v^{(1)}$ and $Q_v^{(2)}$, there is precisely one arrow in $Q_a^{(1)}$ going from the vertex in the set $Q_v^{(1)}$ to the vertex in the set $Q_v^{(2)}$. The same must also be true for each pair of vertices in $Q_v^{(2)}$, $Q_v^{(1)}$, and the arrows in $Q_a^{(2)}$. Hence, if $\alpha \in Q_a^{(1)}$, then $\alpha^* \in Q_a^{(2)}$, and vice versa. If $Q_v^{(1)} = \{a_1, a_2, \dots, a_n\}$, $Q_v^{(2)} = \{b_1, b_2, \dots, b_m\}$ and $\alpha_{b_i a_j} : a_j \to b_i$, $\alpha_{b_i a_j} \in Q_a^{(1)}$, the ensemble can be given by the two matrices

$$\mathcal{A} = \begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline b_1 & \alpha_{b_1a_1} & \alpha_{b_1a_2} & \cdots & \alpha_{b_1a_n} \\ b_2 & \alpha_{b_2a_1} & \alpha_{b_2a_2} & \cdots & \alpha_{b_2a_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_m & \alpha_{b_ma_1} & \alpha_{b_ma_2} & \cdots & \alpha_{b_ma_n} \\ \hline \end{array} \quad \text{and} \quad \mathcal{A}^* = \begin{array}{c|cccc|} & b_1 & b_2 & \cdots & b_m \\ \hline a_1 & \alpha^*_{b_1a_1} & \alpha^*_{b_2a_1} & \cdots & \alpha^*_{b_ma_1} \\ a_2 & \alpha^*_{b_1a_2} & \alpha^*_{b_2a_2} & \cdots & \alpha^*_{b_ma_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & \alpha^*_{b_1a_n} & \alpha^*_{b_2a_n} & \cdots & \alpha^*_{b_ma_n} \\ \hline \end{array}$$

An ensemble Q given by the matrices $\mathcal{A}$ and $\mathcal{A}^*$ and the relation

$$\mathcal{A}^*\mathcal{A} = \begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & \varepsilon_{a_1} & & & \\ a_2 & & \varepsilon_{a_2} & 0 & \\ \vdots & & 0 & \ddots & \\ a_n & & & & \varepsilon_{a_n} \\ \hline \end{array}$$

will be called an ensemble of dimension $m \times n$ with the orthogonality condition

$$\alpha^*_{1i}\alpha_{1j} + \alpha^*_{2i}\alpha_{2j} + \dots + \alpha^*_{mi}\alpha_{mj} = \begin{cases} 0, & \text{if } i \neq j, \\ \varepsilon_{a_i}, & \text{if } i = j \end{cases}$$

defined in the vertices $a_1, a_2, \dots, a_n$. This ensemble will be denoted by $Q_{m \times n^\perp}$.

In this paper, we find conditions for majorization of the $*$-categories, $\mathcal{K}(Q_{k\times l^\perp}) \succ$ $\succ \mathcal{K}(Q_{m\times n^\perp})$ (Theorem 2) and prove a criterion for the $*$-category $\mathcal{K}(Q_{m\times n^\perp})$ to be $*$-wild (Theorem 3). As a tool for proving statements on the majorization, we construct a quiver $Q'_{\alpha,x}$ that is derived with respect to the arrow α at the point $x = (x_1, x_2, \dots, x_n)$, $x_i \in \mathbb{R}$, and prove Theorem 1 related to this construction.

2. CATEGORIES WITH AN INVOLUTION AND A MAJORIZATION RELATION

Let $\mathcal{K}$ be a category with an involution $*$ over the field $\mathbb{C}$ of complex numbers [4, 5] such that $a^* = a$ for every $a \in \operatorname{Ob}\mathcal{K}$. Hence, for every morphism $\alpha\colon a \to b$ there is a morphism $\alpha^*\colon b \to a$ such that 1) $\alpha^{**} = \alpha$, 2) $(\alpha\beta)^* = \beta^*\alpha^*$, 3) $(z_1\alpha_1 + z_2\alpha_2)^* =$ $= \overline{z}_1\alpha_1^* + \overline{z}_2\alpha_2^*$, $z_1, z_2 \in \mathbb{C}$. We assume that there is a zero object in $\mathcal{K}$. In the sequel, we call such categories $*$-categories.

Together with the category $\mathcal{K}$, we will also consider, as a system of generators, an involution quiver ($*$-quiver) Q [4, 5] such that for each vertex a, $a^* = a$, and for each arrow $\alpha\colon a \to b$ there corresponds an arrow $\alpha^*\colon b \to a$ such that $\alpha^{**} = \alpha$. The category $\mathcal{K}$ is obtained from the category of paths of Q by factorization. The category $\mathcal{K}$ will be called finitely generated if it can be defined by a finite $*$-quiver and a finite set of relations (linear combinations of certain paths are set to zero). A morphism $\varphi\colon a \to b$ in the category $\mathcal{K}$ is called an isomorphism if there exists a morphism $\varphi^{-1}\colon b \to a$ such that $\varphi^{-1}\varphi = \varepsilon_a$ and $\varphi\varphi^{-1} = \varepsilon_b$. An isomorphism φ is called a congruence if $\varphi^{-1} = \varphi^*$. A representation of the category $\mathcal{K}$ is an involutive functor π over the field $\mathbb{C}$, compatible with the involution, from the category $\mathcal{K}$ to the category $\mathcal{H}$ of Hilbert spaces, where the objects are separable Hilbert spaces and morphisms are bounded linear operators from one space to another. The involution on the objects of $\mathcal{H}$ is an identity, whereas, it is the adjoint operator on a morphism.

Representations of the category $\mathcal{K}$ themselves form a $*$-category $\operatorname{Rep}\mathcal{K}$ whose objects are involutive functors π ($*$-representations) and morphisms are families of morphisms of the category $\mathcal{H}$ which intertwine these functors (natural transformations of the functors). Two representations are called equivalent, if they are isomorphic in the category of representations, and are unitary isomorphic, if they are congruent in the category of representations. As is known, see for example [4], that if two $*$-representations are equivalent, then they are unitary equivalent.

Let us define a $*$-category of matrices $\mathcal{M}(\mathcal{K})$ over a category $\mathcal{K}$. Its objects are ordered collections $(a_1, a_2, \dots, a_n)$ of objects of the category $\mathcal{K}$ (we do not exclude the possibility of repetitions of a_i). A morphism from $(a_1, a_2, \dots, a_n)$ into $(b_1, b_2, \dots, b_m)$ is a matrix of dimension $m \times n$ that will be written as

$$\mathcal{A} = \begin{array}{c|cccc|}
 & a_1 & a_2 & \cdots & a_n \\ \hline
b_1 & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\
b_2 & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_m & \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \\ \hline
\end{array}\,, \qquad
\mathcal{A}^* = \begin{array}{c|cccc|}
 & b_1 & b_2 & \cdots & b_m \\ \hline
a_1 & \alpha_{11}^* & \alpha_{21}^* & \dots & \alpha_{m1}^* \\
a_2 & \alpha_{12}^* & \alpha_{22}^* & \dots & \alpha_{m2}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & \alpha_{1n}^* & \alpha_{2n}^* & \dots & \alpha_{mn}^* \\ \hline
\end{array}\,,$$

$\alpha_{ij} \in \operatorname{Hom}_{\mathcal{K}}(a_j, a_i)$. The composition and addition of morphisms in the category $\mathcal{M}(\mathcal{K})$ is performed by using the usual matrix multiplication and addition.

Let us give a definition of an enveloping $*$-category, see [1, 2, 3].

Definition 1. *Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be $*$-categories. A pair $(\widetilde{\mathcal{K}}, \Phi\colon \mathcal{K} \to \widetilde{\mathcal{K}})$, where Φ is an involutive functor from the $*$-category $\mathcal{K}$ into the $*$-category $\widetilde{\mathcal{K}}$, is called an enveloping category of the category $\mathcal{K}$ if for any $*$-representation $\pi\colon \mathcal{K} \to \mathcal{H}$ there exists a* unique *representation $\tilde{\pi}\colon \widetilde{\mathcal{K}} \to \mathcal{H}$ such that the diagram*

$$\begin{tikzcd} \widetilde{\mathcal{K}} \arrow[dr, "\tilde{\pi}"] & \\ \mathcal{K} \arrow[u, "\Phi"] \arrow[r, "\pi"] & \mathcal{H} \end{tikzcd}$$

is commutative and any morphism in the category $\operatorname{Rep}\mathcal{K}$ *that intertwines two representations* π_1 *and* π_2 *in the category* $\mathcal{K}$ *intertwines the representations* $\tilde{\pi}_1$ *and* $\tilde{\pi}_2$ *in the category* $\widetilde{\mathcal{K}}$.

Let $\mathcal{M}(\mathcal{K})$ be the category of matrices over a $*$-category $\mathcal{K}$, $\big(\widetilde{\mathcal{M}(\mathcal{K})}, \Phi\colon \mathcal{M}(\mathcal{K}) \to \to \widetilde{\mathcal{M}(\mathcal{K})}\big)$ its enveloping category. Any $*$-representation $\pi\colon \mathcal{K} \to \mathcal{H}$ induces a representation $\Pi\colon \mathcal{M}(\mathcal{K}) \to \mathcal{H}$, hence, a representation $\widetilde{\Pi}\colon \widetilde{\mathcal{M}(\mathcal{K})} \to \mathcal{H}$. If $\Psi\colon \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ is an involutive functor. Then one can naturally construct the functor

$$\mathbf{F}_\Psi\colon \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}.$$

By the definition, $\mathbf{F}_\Psi(\pi) = \widetilde{\Pi} \circ \Psi$ and, if $\mathcal{C} = (\mathcal{C}_a)_{a \in \operatorname{Ob}\mathcal{K}}$ is a morphism of representations in $\operatorname{Rep}\mathcal{K}$, $\mathcal{C}_a\colon \pi(a) \to \pi_1(a)$, $(a_1, a_2, \ldots, a_k) \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$, then [3]

$$\mathbf{F}_\Psi(\mathcal{C}) = \operatorname{diag}(\mathcal{C}_{a_1}, \mathcal{C}_{a_2}, \ldots, \mathcal{C}_{a_k}) \in \operatorname{Hom}_{\mathcal{H}}\big(\pi(a_1) \oplus \cdots \oplus \pi(a_k), \pi_1(a_1) \oplus \cdots \oplus \pi_1(a_k)\big).$$

Definition 2. *We will say that a $*$-category $\mathcal{L}$ directly majorizes a $*$-category $\mathcal{K}$ and denote this by $\mathcal{L} \rhd \mathcal{K}$ if the category of matrices $\mathcal{M}(\mathcal{K})$ contains the enveloping category $(\widetilde{\mathcal{M}(\mathcal{K})}, \mathcal{F})$ and a $*$-functor $\Psi\colon \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ such that the functor $\mathbf{F}_\Psi\colon \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$, which is univalent by the definition, is full. A $*$-category $\mathcal{L}$ majorizes a $*$-category $\mathcal{K}$, denoted by $\mathcal{L} \succ \mathcal{K}$, if there exist $*$-categories $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$ such that $\mathcal{L} \equiv \equiv \mathcal{K}_0 \rhd \mathcal{K}_1 \rhd \cdots \rhd \mathcal{K}_n \rhd \mathcal{K}_{n+1} \equiv \mathcal{K}$. We will say that a $*$-quiver Q_1 majorizes a $*$-quiver Q_2 and denote this by $Q_1 \succ Q_2$, if $\mathcal{K}(Q_1) \succ \mathcal{K}(Q_2)$ for the corresponding categories.*

The majorization relation is a quasi-order on the class of $*$-categories and $*$--quivers.

As a complexity reference for the problem of classifying $*$-representations of $*$--algebras and $*$-categories, we take the classification problem for pairs of self-adjoint operators up to the unitary equivalence [6] or, which is actually the same thing, that for pairs of unitary operators [1, 2, 3].

Definition 3 ([3]). *A $*$-category $\mathcal{K}(Q)$ (a $*$-quiver Q) is called $*$-wild, if $\mathcal{K}(Q) \succ \mathcal{F}_2$, where $\mathcal{F}_2 = \mathbb{C} < u_1, u_2 \mid u_i u_i^* = u_i^* u_i = e,\ i = 1, 2 >$.*

It is clear that if $\mathcal{K}$ is a $*$-wild category and $\mathcal{K}_1 \succ \mathcal{K}$, then $\mathcal{K}_1$ is also $*$-wild. So, it is useful to have a sufficiently large class of "standard" $*$-wild algebras (categories). They include, for example, the $*$-algebra generated by 3 orthogonal projections two of which are mutually orthogonal [1, 2, 6]. In this paper, we construct other "standard" $*$-wild categories.

3. DERIVED $*$-QUIVERS AND MAJORIZATION

Let $\alpha \in Q_a$, $\alpha\colon a \to b$ and, if $a = b$, then $\alpha = \alpha^*$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x_i \neq x_j$ for $i \neq j$ and, in addition, $x_i \geq 0$ if $a \neq b$. If $0 \in \{x_1, x_2, \ldots, x_n\}$, we assume that $x_n = 0$.

For a quiver Q, let us define a quiver $Q'_{\alpha,x}$, *derived with respect to the arrow α at the point x*. To construct $Q'_{\alpha,x}$, we do the following.

1. Delete from the quiver Q the arrows α and $\alpha^*$and replace the vertices a and b with the collections of vertices $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, gluing the vertices a_i and b_i if $x_i \neq 0$.
2. Each self-adjoint arrow $\beta\colon a \to a$ that does not coincide with α is replaced with the ensemble

$$\mathcal{B} = \begin{array}{c|cccc|}
 & a_1 & a_2 & \cdots & a_n \\ \hline
a_1 & \beta_{a_1a_1} & \beta_{a_1a_2} & \cdots & \beta_{a_1a_n} \\
a_2 & \beta_{a_2a_1} & \beta_{a_2a_2} & \cdots & \beta_{a_2a_n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & \beta_{a_na_1} & \beta_{a_na_2} & \cdots & \beta_{a_na_n} \\ \hline
\end{array} = \mathcal{B}^*.$$

   The same is done for each self-adjoint arrow $\beta\colon b \to b$.
3. Each pair of not self-adjoint arrows $\gamma\colon a \to b$ and $\gamma^*\colon b \to a$ is replaced with the ensemble

$$\mathcal{T} = \begin{array}{c|cccc|}
 & a_1 & \cdots & a_{n-1} & a_n \\ \hline
a_1 & \gamma_{a_1a_1} & \cdots & \gamma_{a_1a_{n-1}} & \gamma_{a_1a_n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & \gamma_{a_{n-1}a_1} & \cdots & \gamma_{a_{n-1}a_{n-1}} & \gamma_{a_{n-1}a_n} \\
b_n & \gamma_{b_na_1} & \cdots & \gamma_{b_na_{n-1}} & \gamma_{b_na_n} \\ \hline
\end{array},$$

$$\mathcal{T}^* = \begin{array}{c|cccc|}
 & a_1 & \cdots & a_{n-1} & b_n \\ \hline
a_1 & \gamma^*_{a_1a_1} & \cdots & \gamma^*_{a_{n-1}a_1} & \gamma^*_{b_na_1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & \gamma^*_{a_1a_{n-1}} & \cdots & \gamma^*_{a_{n-1}a_{n-1}} & \gamma^*_{b_na_{n-1}} \\
a_n & \gamma^*_{a_1a_n} & \cdots & \gamma^*_{a_{n-1}a_n} & \gamma^*_{b_na_n} \\ \hline
\end{array}.$$

4. Each pair of arrows $\delta\colon c \to a$, $\delta^*\colon a \to c$, $c \neq a$, $c \neq b$, is replaced with the ensemble

$$\Delta, \qquad \Delta^* = \begin{array}{c|cccc|}
 & a_1 & \cdots & a_{n-1} & a_n \\ \hline
c & \delta^*_{a_1c} & \cdots & \delta^*_{a_{n-1}c} & \delta^*_{a_nc} \\ \hline
\end{array}.$$

   The same is done for each pair of arrows $c \to b$ and $b \to c$, $c \neq a$, $c \neq b$.

The relations for the quiver $Q'_{\alpha,x}$ induced by the replacements are obtained from the relations for the quiver Q in the following way.

In every relation for the quiver Q, replace the morphism α with the morphism matrix

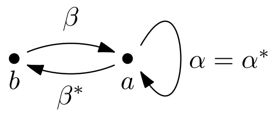
$$\begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & x_1\varepsilon_{a_1} & & & \\ a_2 & & x_2\varepsilon_{a_2} & 0 & \\ \vdots & & 0 & \ddots & \\ a_n & & & & x_n\varepsilon_{a_n} \\ \hline \end{array} \tag{1}$$

if $a = b$ and $\alpha = \alpha^*$. In the case where $\alpha\colon a \to b$ and $a \neq b$, replace α with the morphism matrix (1) if $x_i \neq 0$, $i \in \overline{1,n}$, and with the morphism matrix

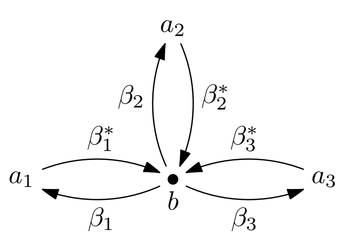
$$\begin{array}{c|ccccc|} & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ \hline a_1 & x_1\varepsilon_{a_1} & & & & \\ a_2 & & x_2\varepsilon_{a_2} & & & \\ \vdots & & & \ddots & & \\ a_{n-1} & & & & x_{n-1}\varepsilon_{a_{n-1}} & \\ b_n & & & & & 0 \\ \hline \end{array}$$

if $x_n = 0$. Also, replace the morphisms $\beta = \beta^*\colon a \to a$, $\tilde{\beta} = \tilde{\beta}^*\colon b \to b$, $\gamma\colon a \to b$, $\gamma^*\colon b \to a$, $\delta\colon c \to a$, $\delta^*\colon a \to c$ ($c \neq a$, $c \neq b$), $\tilde{\delta}\colon d \to b$, $\tilde{\delta}^*\colon b \to d$ ($d \neq a$, $d \neq b$) with the morphism matrices of the corresponding ensembles performing all the operations by formally multiplying and adding the matrices and equating the corresponding elements in the obtained identities. Naturally, one relation gives rise to several relations, in general.

Example 1. *Let, for example, Q be the quiver*



with the relation $\alpha^3 - 6\alpha^2 + 11\alpha - 6\varepsilon_a \equiv (\alpha - \varepsilon_a)(\alpha - 2\varepsilon_a)(\alpha - 3\varepsilon_a) = 0$, $x = (1, 2, 0)$. Then $Q'_{\alpha,x}$ has the form



and the relations are

$$\begin{array}{c|ccc|} & a_1 & a_2 & a_3 \\ \hline a_1 & 0 & & \\ a_2 & & \varepsilon_{a_2} & \\ a_n & & & -\varepsilon_{a_3} \\ \hline \end{array} \cdot \begin{array}{c|ccc|} & a_1 & a_2 & a_3 \\ \hline a_1 & -\varepsilon_{a_1} & & \\ a_2 & & 0 & \\ a_n & & & -2\varepsilon_{a_3} \\ \hline \end{array} \cdot \begin{array}{c|ccc|} & a_1 & a_2 & a_3 \\ \hline a_1 & -2\varepsilon_{a_1} & & \\ a_2 & & -2\varepsilon_{a_2} & \\ a_n & & & -3\varepsilon_{a_3} \\ \hline \end{array} = 0.$$

Hence, $\varepsilon_{a_3} = 0$ and, consequently, a_3 is the zero object. Finally, $Q'_{\alpha,x}$ is the $*$-quiver

$$\begin{tikzcd} a_1 \arrow[r, bend right, "\beta_1"'] & b \arrow[l, bend right, "\beta_1^*"'] \arrow[r, bend left, "\beta_2"] & a_2 \arrow[l, bend left, "\beta_2^*"] \end{tikzcd}$$

with no additional relations.

Remark 1. *In a standard way, a not self-adjoint loop $\gamma\colon a \to a$ and its adjoint γ^* can be replaced with two self-adjoint ones by choosing another system of generators in the $*$-category $\mathcal{K}(Q)$, $\alpha = \frac{\gamma+\gamma^*}{2}$, $\beta = \frac{\gamma-\gamma^*}{2i}$. So, it is always possible to assume that the quiver Q has an arrow α with respect to which one can construct the derived quiver $Q'_{\alpha,x}$.*

Theorem 1. *For a quiver Q, let $Q'_{\alpha,x}$ be the derived quiver with respect to the arrow α at the point $x = (x_1, x_2, \dots, x_n)$. Then $\mathcal{K}(Q) \succ \mathcal{K}(Q'_{\alpha,x})$.*

Proof. Let $\mathcal{M}(Q'_{\alpha,x})$ be the category of matrices over the $*$-category $\mathcal{K}(Q'_{\alpha,x})$. We will prove the claim in the case where $\alpha\colon a \to a$ is a self-adjoint arrow. If $\alpha\colon a \to b$, $a \neq b$, the proof is similar. Let us construct the functor $\Psi\colon \mathcal{K}(Q) \to \mathcal{M}(Q'_{\alpha,x})$ as follows. Set

$$\Psi(\alpha) = \begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & x_1\varepsilon_{a_1} & & & \\ a_2 & & x_2\varepsilon_{a_2} & & \\ \vdots & & & \ddots & \\ a_n & & & & x_n\varepsilon_{a_n} \\ \hline \end{array},$$

$\Psi(\beta) = \beta$ if the arrow is not changed when constructing $Q'_{\alpha,x}$, and set $\Psi(\beta)$ to be the morphism matrix of the ensemble that replaces β, otherwise. Relations for $Q'_{\alpha,x}$ were defined so that the constructed mapping agreed with the relations for Q and $Q'_{\alpha,x}$ and could be extended to a functor from the category $\mathcal{K}(Q)$ into the category $\mathcal{K}(Q'_{\alpha,x})$.

Let us define the functor

$$\mathbf{E}_\Psi\colon \mathrm{Rep}\,(Q'_{\alpha,x}, \mathcal{H}) \to \mathrm{Rep}\,(Q, \mathcal{H})$$

as follows. If $\pi \in \mathrm{Ob}\,\mathrm{Rep}\,(Q'_{\alpha,x}, \mathcal{H})$, then π is extended to a representation Π of the category $\mathcal{M}(Q'_{\alpha,x})$ on $\mathcal{H}$, and we set $\mathbf{F}_\Psi = \Pi \circ \Psi$.

Let $\mathcal{C} = (\mathcal{C}_a)_{a \in Q'_v}$ be a morphism of representations in $\operatorname{Rep}(Q'_{\alpha,x}, \mathcal{H})$, $\mathcal{C}_a \colon \pi(a) \to$ $\to \pi_1(a)$. If $\overline{a} = (a_1, a_2, \ldots, a_k) \in \operatorname{Ob} \mathcal{M}(Q'_{\alpha,x})$, then set

$$\begin{aligned}\big(\mathbf{F}_\Psi(\mathcal{C})\big)_{\overline{a}} &= \operatorname{diag}(\mathcal{C}_{a_1}, \mathcal{C}_{a_2}, \ldots, \mathcal{C}_{a_k}) \\ &\in \operatorname{Hom}_{\mathcal{M}(Q'_{\alpha,x})}\Big(\big(\pi(a_1), \pi(a_2), \ldots, \pi(a_k)\big), \big(\pi_1(a_1), \pi_1(a_2), \ldots, \pi_1(a_k)\big)\Big).\end{aligned}$$

It is easy to check that $\mathbf{F}_\Psi(\mathcal{C})$ is a morphism from the representation $\mathbf{F}_\Psi(\pi)$ into the representation $\mathbf{F}_\Psi(\pi_1)$ (in the category $\operatorname{Rep}(Q, \mathcal{H})$) and $\mathbf{F}_\Psi$ is a functor. It is easy to check as well that, being univalent by the construction, the functor $\mathbf{F}_\Psi$ is also full. Hence, $\mathbf{F}_\Psi$ is an equivalence functor from the category $\operatorname{Rep}(Q'_{\alpha,x}, \mathcal{H})$ into some full subcategory of the category $\operatorname{Rep}(Q, \mathcal{H})$ and, by the definition of the majorization relation, we have $\mathcal{K}(Q) \succ \mathcal{K}(Q'_{\alpha,x})$. $\square$

It is sometimes useful to describe the repeated construction of the derived quiver with respect to arrows in an ensemble $\mathcal{B}$ or a subquiver of a quiver Q as a single construction of the derived quiver $Q'_{\mathcal{B},x}$ with respect to the ensemble $\mathcal{B}$.

Example 2. *Let Q be the ensemble $\mathcal{A}$ and $\mathcal{A}^*$,*

$$\mathcal{A} = \begin{array}{c|cc|} & a_1 & a_2 \\ \cline{2-3} b_1 & \alpha_{11} & \alpha_{12} \\ b_2 & \alpha_{21} & \alpha_{22} \\ \cline{2-3} \end{array}, \qquad \mathcal{A}^* = \begin{array}{c|cc|} & b_1 & b_2 \\ \cline{2-3} a_1 & \alpha^*_{11} & \alpha^*_{21} \\ a_2 & \alpha^*_{12} & \alpha^*_{22} \\ \cline{2-3} \end{array}$$

with the relation

$$\mathcal{A}^*\mathcal{A} = \begin{array}{c|cc|} & a_1 & a_2 \\ \cline{2-3} a_1 & \varepsilon_{a_1} & 0 \\ a_2 & 0 & \varepsilon_{a_2} \\ \cline{2-3} \end{array}.$$

Consider a subensemble $\mathcal{B}$, $\mathcal{B}^*$ in Q, where

$$\mathcal{B}^* = \begin{array}{c|cc|} & b_1 & b_2 \\ \cline{2-3} a_2 & \alpha^*_{12} & \alpha^*_{22} \\ \cline{2-3} \end{array}.$$

Let $Q^{(1)} = Q'_{\mathcal{B},x}$ be the derived quiver for the quiver Q with respect to the subensemble $\mathcal{B}$ at the point x that can be conveniently written as the matrix

$$x = \begin{array}{c|c|}
 & a_2 \\
\hline
b_1 & \begin{matrix} x_1 & & \\ & x_2 & 0 \\ 0 & & \ddots \\ & & & x_m \end{matrix} \\
\hline
b_2 & \begin{matrix} y_1 & & \\ & y_2 & 0 \\ 0 & & \ddots \\ & & & y_m \end{matrix} \\
\hline
\end{array},$$

where $x_i > 0$, $y_i > 0$, $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$, $x_i^2 + y_i^2 = 1$. We wrote the point x as a matrix as to stress that, when constructing $Q'_{\mathcal{B},x}$, the ensemble $\mathcal{B}$, $\mathcal{B}^*$ is replaced with the matrix

$$\begin{array}{c|cccc|}
 & a_2^{(1)} & a_2^{(2)} & \dots & a_2^{(m)} \\
\hline
a_2^{(1)} & x_1\varepsilon_{a_2^{(1)}} & & & \\
a_2^{(2)} & & x_2\varepsilon_{a_2^{(2)}} & & 0 \\
\vdots & & 0 & \ddots & \\
a_2^{(m)} & & & & x_m\varepsilon_{a_2^{(m)}} \\
\hline
a_2^{(1)} & y_1\varepsilon_{a_2^{(1)}} & & & \\
a_2^{(2)} & & y_2\varepsilon_{a_2^{(2)}} & & 0 \\
\vdots & & 0 & \ddots & \\
a_2^{(m)} & & & & y_m\varepsilon_{a_2^{(m)}} \\
\hline
\end{array}$$

and its adjoint.

The subensemble

$$\widetilde{\mathcal{A}}^* = \begin{array}{c|cc|}
 & b_1 & b_2 \\
\hline
a_1 & \alpha_{11}^* & \alpha_{12}^* \\
\hline
\end{array}, \qquad \widetilde{\mathcal{A}}$$

in the ensemble $\mathcal{A}^*$, $\mathcal{A}$ will be replaced with

$$\mathcal{A}^{(1)*} = \begin{array}{c|ccc|ccc|} & a_2^{(1)} & a_2^{(2)} \; \ldots & a_2^{(m)} & a_2^{(1)} & a_2^{(2)} \; \ldots & a_2^{(m)} \\ \hline a_1 & \alpha_{11}^{(1)*} & \alpha_{11}^{(2)*} \; \ldots & \alpha_{11}^{(m)*} & \alpha_{11}^{(1)*} & \alpha_{11}^{(2)*} \; \ldots & \alpha_{11}^{(m)*} \\ \hline \end{array}, \qquad \mathcal{A}^{(1)}.$$

The relations in the quiver Q imply that

$$x_i\alpha_{11}^{(i)} + y_i\alpha_{21}^{(i)} = 0, \quad \sum_{i=1}^{m}(\alpha_{11}^{(1)*}\alpha_{11}^{(i)} + \alpha_{21}^{(i)*}\alpha_{21}^{(i)}) = \varepsilon_{a_1}$$

or, since $\alpha_{21}^{(i)} = -\frac{x_i}{y_i}\alpha_{11}^{(i)}$, we get

$$\sum_{i=1}^{m}\frac{\alpha_{11}^{(i)*}\alpha_{11}^{(i)}}{y_i^2} = \varepsilon_{a_1}.$$

If the system of generators in the category $\mathcal{K}(Q^{(1)})$ is changed by replacing the arrows $\alpha_{11}^{(i)}$ with $\beta_i = \frac{\alpha_{11}^{(i)}}{y_i}$, then we get that the category $\mathcal{K}(Q^{(1)})$ is generated by the ensemble

$$\mathcal{A}^{(2)*} = \begin{array}{c|ccc|} & a_2^{(1)} & a_2^{(2)} \; \ldots & a_2^{(m)} \\ \hline a_1 & \beta_{11}^{(1)*} & \beta_{11}^{(2)*} \; \ldots & \beta_{11}^{(m)*} \\ \hline \end{array}, \qquad \mathcal{A}^{(2)}$$

and the relation $\mathcal{A}^{(2)*} \cdot \mathcal{A}^{(2)} = \varepsilon_{a_1}$.

4. ENSEMBLES WITH AN ORTHOGONALITY CONDITION

We will call an ensemble

$$\mathcal{A} = \begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline b_1 & \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ b_2 & \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_m & \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \\ \hline \end{array}, \qquad \mathcal{A}^*$$

with the relation

$$\mathcal{A}^*\mathcal{A} = \begin{array}{c|cccc|} & a_1 & a_2 & \cdots & a_n \\ \hline a_1 & \varepsilon_{a_1} & & & \\ a_2 & & \varepsilon_{a_2} & 0 & \\ \vdots & & 0 & \ddots & \\ a_n & & & & \varepsilon_{a_n} \\ \hline \end{array}, \qquad \alpha_{ij}\colon a_j \to b_i,$$

an ensemble of dimension $m \times n$ with the orthogonality condition

$$\alpha^*_{1i}\alpha_{1j} + \alpha^*_{2i}\alpha_{2j} + \dots + \alpha^*_{mi}\alpha_{mj} = \begin{cases} 0, & \text{if} \quad i \neq j, \\ \varepsilon_{a_i}, & \text{if} \quad i = j \end{cases}$$

in the vertices $a_1, a_2, \dots, a_n$ and denote it by $Q_{m\times n^\perp}$.

Theorem 2. *The following relations take place for the categories:*

1. $\mathcal{K}(Q_{k\times l^\perp}) \succ \mathcal{K}(Q_{m\times n^\perp})$ *for* $k \geq 2$, $l \geq 2$, *and any* $m, n \in \mathbb{N}$;
2. $\mathcal{K}(Q_{k\times 1^\perp}) \succ \mathcal{K}(Q_{m\times n^\perp})$ *for* $k \geq 3$ *and any* $m, n \in \mathbb{N}$.

Proof. Since the majorization relation is transitive and $\mathcal{K}(Q_{m_1\times n_1{}^\perp}) \succ \mathcal{K}(Q_{m_2\times n_2{}^\perp})$ for $m_1 \geq m_2$ and $n_1 \geq n_2$, it remains to prove that

a) $\mathcal{K}(Q_{2\times 2^\perp}) \succ \mathcal{K}(Q_{m\times n^\perp})$;

b) $\mathcal{K}(Q_{(m+1)\times 1^\perp}) \succ \mathcal{K}(Q_{m\times n^\perp})$;

c) $\mathcal{K}(Q_{3\times 1^\perp}) \succ \mathcal{K}(Q_{2\times 2^\perp})$.

Ad a) Follows from Example 2 and Theorem 1.
Ad b) Suppose we have an ensemble $\mathcal{K}(Q_{(m+1)\times 1^\perp})$ given by the matrix

$$\mathcal{A}^* = \begin{array}{c|ccc|} & b_1 & b_2 \ \dots & b_{m+1} \\ \hline a & \alpha^*_1 & \alpha^*_2 \ \dots & \alpha^*_{m+1} \\ \hline \end{array}, \qquad \mathcal{A}$$

and the relation $\mathcal{A}^*\mathcal{A} = \varepsilon_a$. Construct the derived quiver $Q' = Q'_{\alpha_{m+1},x}$, where $x = (x_1, x_2, \dots, x_n)$, $x_i \neq x_j$ for $i \neq j$, $0 < x_i < 1$. As a result we get the ensemble

$$\mathcal{B} = \begin{array}{c|ccc|} & a_1 & \cdots & a_n \\ \hline b_1 & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & \alpha_{m1} & \dots & \alpha_{mn} \\ \hline \end{array}, \qquad \mathcal{B}^*$$

with the relation

$$\mathcal{B}^*\mathcal{B} = \begin{array}{c|ccc|} & a_1 & \cdots & a_n \\ \hline a_1 & (1-x_1^2)\varepsilon_{a_1} & & \\ \vdots & & \ddots & 0 \\ & 0 & & \\ a_n & & & (1-x_n^2)\varepsilon_{a_n} \\ \hline \end{array}.$$

By replacing, in the category $\mathcal{K}(Q')$, the generators α_{ij} with $\beta_{ij} = \alpha_{ij}/\sqrt{1-x_j^2}$, we get the ensemble $Q_{m\times n^\perp}$. Consequently, by Theorem 1, we see that $\mathcal{K}(Q_{(m+1)\times 1^\perp}) \succ \succ \mathcal{K}(Q_{m\times n^\perp})$.

Ad c) Let $Q = Q_{3\times 1^\perp}$. As in b), construct the derived quiver $Q'_{\alpha_3,x}$, where $x = (x_1, x_2)$, $x_1 \neq x_2$, $0 < x_i < 1$. We see that $\mathcal{K}(Q') = \mathcal{K}(Q_{2\times 2^\perp})$ and, consequently, $\mathcal{K}(Q_{3\times 1^\perp}) \succ \mathcal{K}(Q_{2\times 2^\perp})$ by Theorem 1. □

We also have the following theorem.

Theorem 3. *The category $\mathcal{K}(Q_{k\times l^\perp})$ is $*$-wild if and only if $k \geq 2$ and $l \geq 2$, or $k \geq 3$.*

Proof. The proof is actually given in [1] when proving Theorem 4.

Let $\mathcal{F}_2 = \mathbb{C} < u_1, u_2 \mid u_i u_i^* = u_i^* u_i = e >$. Let us show that $\mathcal{K}(Q_{3\times 1^\perp}) \succ C^*(\mathcal{F}_2)$. Suppose $\mathcal{K}(Q_{3\times 1^\perp})$ is given by the ensemble $\mathcal{A}^*$, $\mathcal{A}$ with

$$\mathcal{A}^* = \begin{array}{c|ccc|} & b_1 & b_2 & b_3 \\ \hline a & \alpha_1^* & \alpha_2^* & \alpha_3^* \\ \hline \end{array}.$$

Set

$$\Psi(\alpha_1) = \begin{array}{c|ccccc|} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline b_1^{(1)} & e & 0 & 0 & 0 & 0 \\ b_1^{(2)} & 0 & e & 0 & 0 & 0 \\ b_1^{(3)} & 0 & 0 & 2e & 0 & 0 \\ b_1^{(4)} & 0 & 0 & 0 & 3e & 0 \\ \hline \end{array} \cdot \frac{1}{N}, \quad \Psi(\alpha_2) = \begin{array}{c|ccccc|} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline b_2^{(1)} & e & 0 & e & e & e \\ b_2^{(2)} & 0 & 2e & e & u_1 & 0 \\ b_2^{(3)} & 0 & 0 & e & 0 & u_2 \\ \hline \end{array} \cdot \frac{1}{N},$$

$$\Psi(\alpha_3) = \sqrt{\mathrm{I} - \Psi(\alpha_1)^*\Psi(\alpha_1) - \Psi(\alpha_2)^*\Psi(\alpha_2)},$$

where $\mathrm{I} = \mathrm{diag}\,\{e, e, e, e, e\}$ and N is chosen so that $\|\Psi(\alpha_1)^*\Psi(\alpha_1) - \Psi(\alpha_2)^*\Psi(\alpha_2)\| < < 1$ in the matrix algebra $\mathcal{M}_5\big(C^*(\mathcal{F}_2)\big)$. It can be directly checked that the functor $\mathbf{F}_\Psi\colon \mathrm{Rep}\,\big(C^*(\mathcal{F}_2)\big) \to \mathrm{Rep}\,(Q_{3\times 1^\perp})$ is univalent and full. Finally, the proof of the theorem follows from Theorem 2. □

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